EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS
Vol. 14, No. 3, 2021, 989-1001
ISSN 1307-5543 – ejpam.com
Published by New York Business Global

On Gamma LA-Rings and Gamma LA-Semirings

Waheed Ahmad Khan[1], Abdelghani Taouti[2,*], Azar Salami[2], Zahid Hussain[1]

[1] *Department of Mathematics, University of Education Lahore Attock Campus, Pakistan*
[2] *ETS-Maths and NS Engineering Division, HCT, University City P. O. Box 7947, Sharjah, United Arab Emirates*

Abstract. In this note, first we add some new results in Gamma LA-rings and then we initiate the notion of Γ-LA-semirings. Moreover, we introduce and discuss the terms left ideals, right ideals, bi-ideal, quasi ideals, almost prime and weakly almost prime ideals of a Γ-LA-semiring and their characterizations.

2020 Mathematics Subject Classifications: 17D20, 16Y60

Key Words and Phrases: Gamma LA-Rings, Gamma LA-Semirings, quasi ideals of a Gamma-LA-semiring, weakly almost prime ideals of a Gamma-LA-semiring

---

1. Introduction

Γ-ring was introduced by N. Nobusawa in [13] as a generalization of classical rings. *Γ*-rings have also viewed as the connection with the abelian additive groups of all linear mappings between two finite dimensional spaces over a field. Classical example of *Γ*-ring presented by Nobusawa was by taking an additive group M consisting of homomorphisms of a module A to a module B and an additive group Γ consisting of homomorphisms of B to A, and $a\alpha b$ the usual composite map, where $a, b \in M$ and $\alpha \in \Gamma$. Barnes introduced radical theory of Γ-rings in [1]. Afterwards, numbers of researchers have been published their research articles on Γ-rings. Similarly, Γ-nearrings were introduced by Satyanarayana in [17]. Booth et al. provided different ways to construct equiprime Γ-nearrings [2]. Γ-semirings were introduced by Rao in [14]. Prime and semi-prime ideals of of Γ-semirings were discussed in [5, 6]. Moreover, Quasi-ideals in Γ-semiring were discussed in [7, 8]. The properties of ideals, prime ideals, semi-prime ideals and their generalization plays a key role in structure of Γ-semirings. However, the properties of an ideal in semirings and Γ-semirings are slightly changed from the properties of the usual ring ideals. Theory of ideals in an ordered Γ-semiring have been introduced in [15]. Similarly, weakly prime and weakly primary ideals in gamma seminearrings have been introduced in [9].

---

*Corresponding author.
DOI: https://doi.org/10.29020/nybg.ejpam.v14i3.4034

Email addresses: sirwak2003@yahoo.com (W. A. Khan), ganitaouti@yahoo.com.au (Ab. Taouti), asalami@hct.ac.ae (A. Salami), zahidbsc449@gmail.com (Z. Hussain)

A groupoid which satisfies the left invertive law i.e., $(xy)z = (zy)x$ is said to be an LA-groupoid. A groupoid satisfying the medial law i.e., $(xy)(zt) = (xz)(yt)$ holds by groupoid is called medial [3], whereas a groupoid which satisfying the paramedial law i.e., $(st)(uv) = (vt)(us)$ is a paramedial. LA-groupoid S always obeys medial law, whereas paramedial law holds only by LA-groupoid S with left identity e [3]. LA-groupoid S having e as a left identity holds $p(qr) = q(pr)$ [12], $a \in S$ is left (right) cancellative if $al = am \Rightarrow l = m$ $(la = ma \Rightarrow l = m)$ $\forall\ l, m \in S$. If every element is left and right cancellative then S is cancellative and $x \in S$ is cancellative if x is left and right cancellative. The notion LA-groupoid to LA-group was extended by Kamran [16]. Similarly, if e is left identity in LA-groupoid (i-e $em = m\ \forall\ \ m \in S$) and $\forall\ m \in S\ \exists\ m^{-1} \in S$ such that $m^{-1}m = mm^{-1} = e$, then S is called LA-group. LA-semirings are developed by the concepts of LA-semigroup [10, 11]. LA-semiring and certain results on LA-semirings having two variables are described in [4]. A nonempty set R with two binary operation "." and "+" such that (i) $(R, +)$ is LA-group (ii) $(R, \cdot)$ is LA-groupoid, and non- associative structure w.r.t $'+'$ and $'\cdot'$ satisfying left and right distributive laws is called LA-ring [20]. LA-ring was further elaborated in [18]. Every $x \neq 0$ element of left almost ring R has multiplicative inverse x^{-1} and having left identity e then LA-ring R is called LA-field. LA-ring $< R,\ \oplus, . >$ can be obtain by defining $p \oplus q = q - p$ and pq, for $p, q, r \in R$, is similar as in the ring. The addition in LA-ring cannot assume to be commutative. If for $p, q \in R$, $pq = 0$ implies $p = 0$ or $q = 0$ then LA-ring R is called LA-integral domain. If $\emptyset \neq S \subseteq R$ and S is LA-ring under binary operation defined in R, then S is LA-subring. If $RS \subseteq S$, then S is left ideal of R. Similarly we can define right and two-sided ideals. If $PQ \in A \Longrightarrow P \in A$ or $Q \in A$ then ideal A of R is called prime. Left primary and weakly left primary ideals in Γ-LA-rings and their characterizations are presented in [19].

It is well known that an ideal I of a semiring R is called subtractive, if whenever $a, a+b \in I$, bR, we have $b \in I$. Similarly, A left k-ideal I of a semiring S is a left ideal such that if $a \in$ and $x \in S$ and if either $a + x \in I$. or $x + a \in I$, then $x \in I$.

In this note, first we add few new theorems and examples in the theory of Γ-LA-rings and then we introduce the notion of Γ-LA-semirings. In due course, we describe c-prime, 3-prime ideals and their relationships among themselves in Γ-LA-ring and Γ-LA-semirings. Finally, we discuss left ideals, right ideals, and some results on bi-ideal, quasi ideals, almost prime and weakly almost prime ideals in Γ-LA-semiring.

2. Main Results and Discussions

2.1. Some applications of prime ideals in Γ-La-ring

In this section, we introduce different types of prime ideals in Γ-LA-rings along with their applications. We begin by recalling definition of Γ-LA-ring and then we add few new results and examples in the theory of Γ-LA-ring.

Definition 1. *[19] Let* $(R, +)$ *and* $(\Gamma, +)$ *be the two LA-groups and there exists a mapping* $R \times \Gamma \times R \to R$ *by* $(a, \alpha, b) \to a\alpha b$, *for all* $a, b \in R$ *and* $\alpha \in \Gamma$ *is called a gamma LA-ring, if it satifies the following conditions.*

1. $a\alpha(b+c) = a\alpha b + a\alpha c$
2. $(a+b)\alpha c = a\alpha c + b\alpha c$
3. $a(\alpha+\beta)b = a\alpha b + a\beta b$
4. $(a\alpha b)\beta c = (c\alpha b)\beta a, \forall\ a, .b, c \in R\ , \alpha, \beta \in \Gamma$

Example 1. *Let* $R = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ *be a set with two binary operations "+" and "." given in the Tables set 1 be the LA-ring and* $\Gamma = \{s_1, s_2, s_3\}$ *with binary operation* $\bigoplus$ *is LA-group.*

Tables Set 1

$+$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_2	a_3	a_1	a_4	a_2	a_7	a_5	a_8	a_6
a_3	a_2	a_4	a_1	a_3	a_6	a_8	a_5	a_7
a_4	a_4	a_3	a_2	a_1	a_8	a_7	a_6	a_5
a_5	a_5	a_6	a_7	a_8	a_1	a_2	a_3	a_4
a_6	a_7	a_5	a_8	a_6	a_3	a_1	a_4	a_2
a_7	a_6	a_8	a_5	a_7	a_2	a_4	a_1	a_3
a_8	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1

$\cdot$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_1	a_5	a_5	a_1	a_1	a_5	a_5	a_1
a_3	a_1	a_5	a_5	a_1	a_1	a_5	a_5	a_1
a_4	a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1
a_5	a_1	a_4	a_4	a_1	a_1	a_4	a_4	a_1
a_6	a_1	a_8	a_8	a_1	a_1	a_8	a_8	a_1
a_7	a_1	a_8	a_8	a_1	a_1	a_8	a_8	a_1
a_8	a_1	a_4	a_4	a_1	a_1	a_4	a_4	a_1

$\bigoplus$	s_1	s_2	s_3
s_1	s_1	s_2	s_3
s_2	s_3	s_1	s_2
s_3	s_2	s_3	s_1

Then, clearly R *is* Γ*-LA-ring under operation* $x\gamma y = xy$ *where* $x, y \in R$ *and*$\gamma \in \Gamma$.

Example 2. *Let* $R = \{k, l, m, n, o, p, q, r\}$ *with two binary operations "+" and "." given in Tables set 2 be the LA-ring and* $\Gamma = \{s, t, u, v, w\}$ *with binary operation* $\bigoplus$ *is LA-group.*

Tables Set 2

$+$	k	l	m	n	o	p	q	r
k	m	k	n	l	q	r	o	p
l	n	m	l	k	r	q	p	o
m	k	l	m	n	o	p	q	r
n	l	n	k	m	p	o	r	q
o	q	r	o	p	m	k	n	l
p	r	q	p	o	n	m	l	k
q	o	p	q	r	k	l	m	n
r	p	o	r	q	l	n	k	m

$\cdot$	k	l	m	n	o	p	q	r
k	k	k	k	k	k	k	k	k
l	k	m	k	m	k	k	m	m
m	k	o	k	o	k	k	o	o
n	k	p	k	p	k	k	p	p
o	k	k	k	k	k	k	k	k
p	k	o	k	o	k	k	o	o
q	k	m	k	m	k	k	m	m
r	k	p	k	p	k	k	p	p

$\bigoplus$	s	t	u	v	w
s	s	t	u	v	w
t	w	s	t	u	v
u	v	w	s	t	u
v	u	v	w	s	t
w	t	u	v	w	s

Then R is Γ-LA-ring under operation $x\gamma y = x + \gamma + y$ where $x, y \in R$ and $\gamma \in \Gamma$

Definition 2. *An ideal I is said to be a c-prime ideal of a Γ-LA-ring R, if $x, y \in R$, $\gamma \in \Gamma$ and $x\gamma y \in I \Longrightarrow x \in I$ or $y \in I$.*

Definition 3. *An ideal I is said to be a 3-prime ideal of a Γ-LA-ring R if x, $y \in R$, $\beta \in \Gamma$ and $x\beta s\beta y \in I$ for all $s \in R$ implies $x \in I$ or $y \in I$.*

We present few relationships among c-prime, 3-prime and prime ideals.

Lemma 1. *In a Γ-LA-ring R, every c-prime ideal is a 3-prime ideal.*

Proof. Let I be a c-prime ideal of Γ-LA-ring R. Let $x, y \in R$, $\beta \in \Gamma$ and $x\beta s\beta y \in I$ for all $s \in R$. As I is c-prime ideal so $x \in I$ or $y \in I$. So I is a 3-prime ideal of R.

Lemma 2. *In a Γ-LA-ring R, every 3-prime ideal is a prime ideal.*

Proof. Let I be a 3-prime ideal of Γ-LA-ring R. Let $y \in I$, $\gamma \in \Gamma$ and $x\gamma y \in I$. As I is 3-prime ideal so $x \in I$ or $y \in I$. Clearly, I is prime ideal of R.

Lemma 3. *Every c-prime ideal in Γ-LA-ring is a prime ideal.*

Proof. Let I be a c-prime ideal of Γ-LA-ring R, and let $y \in I$, $\gamma \in \Gamma$ and $x\gamma y \in I$. Since I is c-prime ideal, $x \in I$ or $y \in I$. Then I is a prime ideal of R.

Theorem 1. *Let R be a Γ-LA-ring. Then, I is a 3-prime ideal in R iff R/I is a Γ-LA-integral domain.*

Proof. ($\Longrightarrow$)Let I be a 3-prime ideal of R, then by Lemma 2, I is a prime ideal. Thus, R/I is a Γ-LA-integral domain.
($\Longleftarrow$) Suppose that R/I is a Γ-LA-integral domain with $x\beta s\beta y \in I$ for all $s \in R$. Then $I + arb = I$, so $(I + a)(I + b) = I$, where $a, b \in R$. Since R/I is Γ -LA-integral domain, we have $I + a = I$ or $I + b = I$, then aI *or* bI is in I. Hence I is a 3-prime ideal of R.

Theorem 2. *Let I be an ideal of a Γ -LA-ring R. Then, I is a c-prime ideal in R iff R/I is a Γ-LA-integral domain.*

Proof. (⇒) Let I be a c-prime ideal in R. Then I is a prime ideal by Lemma 3. Thus R/I is a Γ-LA-integral domain.
(⇐) Assume that R/I is a Γ-LA-integral domain with $x\beta s\beta y \in I$ for all $s \in R$. Then, $I+arb = I$ so $(I+a)(I+b) = I$, $a, b \in R$. Since R/I is a Γ-LA-integral domain, $I+a = I$ or $I+b = I$, then aI *or* bI is in I. Thus I is a c-prime ideal of R.

Here we introduce the notion of weakly prime ideal in Γ-LA-rings.

Definition 4. *A proper ideal I is said to be a weakly prime ideal of a Γ-LA-ring R if $0 \neq A\Gamma B \subseteq I$ implies either $A \subseteq I$ or $B \subseteq I$ for any ideals A and B of R.*

Remark 1. *Obviously every prime ideal is weakly prime and $\{0\}$ is always weakly prime ideal.*

Theorem 3. *Let I be a weakly prime ideal of* Γ*-LA-ring which is not prime. Then $I = 0$.*

Proof. Since I is a weakly prime (but not prime), there exist ideals $A \nsubseteq I$ and $B \nsubseteq I$ but $0 = A\Gamma B \subseteq I$. Since $I \subseteq A+I$ and $B \subseteq B+I$. But, if $I^2 \neq 0$, by distributive laws "**.**" over " + " of Γ-LA-ring, we have

$$0 \neq I^2 = I\Gamma I \subseteq (A+I)\Gamma(B+I)$$

$$= [(A+I)\Gamma B] + [(A+I)\Gamma I]$$

$$= A\Gamma B + I\Gamma B + A\Gamma I + I\Gamma I$$

$$\subseteq I.$$

Which implies $(A+I) \subseteq I$ and$(B+I) \subseteq I$, since I is a weakly prime i.e., $A \subseteq I$ or $B+I \subseteq I$, a contradiction. Hence, $I^2 = 0$.

Remark 2. *It is clear that if $R^2 = R\Gamma R = 0$ then every ideal of gamma left almost ring is a weakly prime.*

Theorem 4. *In gamma left almost ring R, every ideal is a weakly prime iff $A\Gamma B = A$, $A\Gamma B = B$ or $A\Gamma B = 0$, for any ideals A, B of R.*

Proof. Assume that every ideal in R is a weakly prime ideal. Let A, B be weakly prime ideals of R**,** then $A\Gamma B$ is a left ideal of R provided that $AB{\neq}R,$ then by hypothesis, $A\Gamma B$ is weakly prime. We consider two situations, that is $A\Gamma B = 0$ or $A\Gamma B \neq 0$. If $0 \neq A\Gamma B \subseteq AB$, then by Definition 4 we have $A \subseteq A\Gamma B$ or $B \subseteq A\Gamma B$. Since A and B are ideals of R, we have $A\Gamma B \subseteq A$ and $A\Gamma B \subseteq B$. Therefore, $A = A\Gamma B$ or $B = A\Gamma B$. If $A\Gamma B = R$ then $A = B = R$, whence $R^2 = R$. Conversely, for proper ideal I of R and ideals A and $B,$ suppose that $0 \neq A\Gamma B \subseteq I$. Then either $A = A\Gamma B \subseteq I$ or $B{=}\ A\Gamma B \subseteq I$.

Corollary 1. *If every ideal of gamma left almost ring is weakly prime. Then, either $A^2 = A$ or $A^2 = 0$ for ideal A.*

Theorem 5. *In Γ-LA-ring every c-prime ideal is a weakly prime ideal.*

Proof. Let I be c-prime ideal of Γ-LA-ring R, by Lemma 1, I is a prime ideal. So I is a weakly prime ideal of R because every prime ideal is weakly prime ideal.

Theorem 6. *Every 3-prime ideal in Γ-LA-ring is a weakly prime ideal.*

Proof. Let I is 3-prime ideal of Γ-LA-ring R, by Lemma 2, we have I is a prime ideal. Since every prime ideal is a weakly prime ideal, we have I a weakly prime ideal of R.

3. Γ-LA-SEMIRINGS

In this section, we introduce the notion of Γ-LA-semiring. Furthermore, we introduce prime, weakly prime, subtracted ideals and nilpotent elements in Γ-LA-semiring along with some interesting results.

Definition 5. *Let $(S,+)$ and $(\Gamma,+)$ be the two LA-monoids. Then S is said to be a gamma LA-semiring (or Γ-LA-semiring) if there exists a mapping $S \times \Gamma \times S \to S$ written (x,γ,y) by $x\gamma y$ such that the following axioms hold*
1. $x\gamma(y+z) = x\gamma y + x\gamma z$ and $(x+y)\gamma z = x\gamma z + y\gamma z$
2. $x(\gamma+\beta)y = x\gamma y + x\beta y$
3. $(x\gamma y)\beta z = (z\gamma y)\beta x$
for all $x,y,z \in S$, $\gamma,\beta \in \Gamma$. In this case we denote Γ-LA-semiring by $(S,\ \Gamma)$.

Example 3. *Let $S = \{a,b,c,d,e\}$ with two binary operations " $+$ " and " $.$" given in the Tables set 3 be the LA-semiring and $\Gamma = \{p,q,r,s\}$ with binary operation "$\bigoplus$" is LA-monoid.*

Tables Set 3

$+$	a	b	c	d	e
a	s	t	u	v	w
b	a	a	d	a	b
c	a	b	b	d	e
d	a	a	b	a	d
e	a	d	e	b	c

$\cdot$	a	b	c	d	e
a	a	b	c	d	e
b	e	a	b	c	d
c	d	e	a	b	c
d	c	d	e	a	b
e	b	c	d	e	a

$\bigoplus$	p	q	r	s
p	p	p	p	p
q	p	q	r	s
r	p	s	q	r
s	p	r	s	q

Then, S is LA- Γ -semiring under operations, $(x\gamma y)\beta z = (z\gamma y)\beta x$ and $x\gamma y = xy$ where $x,y \in S$ and $\gamma \in \Gamma$.

Definition 6. *Let I be a proper ideal of gamma LA-semiring S and $AB \subseteq I$ such that $A \subseteq I$ or $B \subseteq I$ for any ideals A,B of S, then proper ideal I of a gamma LA-semiring S is a prime ideal.*

Definition 7. *If I is a proper ideal of Γ-LA-semiring S and $\{0\} \neq A\Gamma B \subseteq I$ such that $A \subseteq I$ or $B \subseteq I$ for any ideals A,B of S, then I is called weakly prime ideal of gamma LA-semiring S.*

Definition 8. *If $sn = 0$ for $s \in S$ and positive integer n (depending on s), then the element s in a Γ-LA-semiring S is nilpotent. The set of all nilpotent element of S is denoted by Nil S.*

Definition 9. *If $In = 0$ for positive integer n (depending on I), then ideal I in a Γ-LA-semiring S is nilpotent.*

Theorem 7. *Let A be a subtractive ideal in a Γ-LA-semiring S with $1 \neq 0$. Then the followings are equivalent.*
(i) A is a weakly prime ideal.
(ii) If $\{0\} \neq X\Gamma Y \subseteq A$ for right (left) ideals X, Y of S, then $X \subseteq A$ or $Y \subseteq A$.
(iii) If $x, y \in S$ such that $\{0\} \neq x\Gamma S\Gamma y \subseteq A$, then $x \in A$ or $y \in A$.

Proof. $(i) \Rightarrow (ii)$ Let A be a weakly prime ideal of S and X, Y are two right (left) ideals of S such that $\{0\} \neq X\Gamma Y \subseteq A$. Let the ideals generated by X, Y are $< X >, < Y >$, respectively. Then $\{0\} \neq < X > \Gamma < Y > \subseteq A$ implies $< X > \subseteq A$ or $< Y > \subseteq A$ and $X \subseteq < X > \subseteq A$ or $Y \subseteq < Y > \subseteq A$. Therefore, $X \subseteq A$ or $Y \subseteq A$.
$(ii) \Rightarrow (iii)$ Let $\{0\} \neq x\Gamma S\Gamma y \subseteq A$. Since S has an identity, therefore $\{0\} \neq (x\Gamma S)(y\Gamma S) \subseteq A$ implies $x \in x\Gamma S \subseteq A$ or $y \in y\Gamma S \subseteq A$.
$(iii) \Rightarrow (i)$ Suppose that $X\Gamma Y \subseteq A$, for ideals X and Y of S, where $X \nsubseteq A$ and $Y \nsubseteq A$. Let $x \in X\backslash A$, $y \in Y\backslash A$. Also let $x' \in XnA$, $y' \in YnA$ be chosen arbitrary. Since $x + x', y + y' \notin A$, we must have $\{0\} = (x + x')\Gamma S\Gamma(y + y')$. Now if we are letting $x' = 0$ or $y' = 0$ or $x' = 0$ and $y' = 0$ and considering all combinations we get $0 = x\gamma y = x'\gamma y = x\gamma y' = x'\gamma y'$ and hence $X\Gamma Y = \{0\}$.

Proposition 1. *Every ideal of a gamma LA-semiring S is weakly prime iff we have $X\Gamma Y = X$, $X\Gamma Y = Y$, or $X\Gamma Y = 0$, for any ideals X, Y in S.*

Proof. Assume X and Y are the weakly prime ideals of S. Suppose $X\Gamma Y \neq S$. Then $X\Gamma Y$ is a weakly prime. If $\{0\} \neq X\Gamma Y \subseteq X\Gamma Y$, then we have $X \subseteq X\Gamma Y$ or $Y \subseteq X\Gamma Y$ (since $X\Gamma Y$ is weakly prime ideal of S), that is, $X = X\Gamma Y$ or $Y = X\Gamma Y$. If $X\Gamma Y = S$ then we have $X = Y = S$, whence $SAs = S$.
Conversely, let A be any proper ideal of S and let $\{0\} \neq X\Gamma Y \subseteq A$ for ideals X and Y of S. Then, we have either $X = X\Gamma Y \subseteq A$ or $Y = X\Gamma Y \subseteq A$.

On the basis of above proposition we can easily prove the following results.

Remark 3. *If every ideal of Γ-LA-semiring S is a weakly prime, then we have either $X^2 = X$ or $X^2 = 0$, for any ideal X of S.*

Lemma 4. *Let P be a subtractive and weakly prime ideal but not a prime ideal of Γ-LA-semiring S. Let $x\gamma y = 0$, for some $x, y \notin P$, then we have $x\Gamma P = P\Gamma y = \{0\}$.*

Proof. Suppose $x\gamma p_1 \neq 0$, for some $p_1 \in P$ and $\gamma \in \Gamma$. Then $0 \neq x\gamma(y + p_1) \in P$. Since P is a weakly prime ideal of S, therefore $y + p_1 \in P$ or $x \in P$, that is, $x \in P$ or $y \in P$, a contradiction. Therefore $x\Gamma P = \{0\}$. Similarly, we can show that $P\Gamma y = \{0\}$.

Theorem 8. *Let P be a subtractive ideal of a Γ-LA-semiring S. If P is weakly prime but not a prime, then $P^2 = \{0\}$.*

Proof. Suppose $p_1\gamma p_2 \neq 0$, for some $p_1, p_2 \in P$ and $\gamma \in \Gamma$ and $x\gamma y = 0$, for some $x, y \notin P$, where P is not a prime ideal of S. Then by Lemma 4 we have $(x+p_1)\gamma(y+p2) = p_1\gamma p_2 \neq 0$. Hence either $(x + p_1) \in P$ or $(y + p_2) \in P$, and thus either $x \in P$ or $y \in P$, a contradiction. Hence $P^2 = \{0\}$.

3.1. IDEALS IN Γ-LA-SEMIRING

In this section, we introduce the left and right ideals of Γ-LA-semiring and present some results on bi-ideal and quasi ideal in Γ-LA-semiring.

Lemma 5. *Let S be a gamma LA-semiring with identity. Then $a\gamma b = a\beta b$, for all $a, b \in S$ and $\gamma, \beta \in \Gamma$.*

Proof. Let S be a Γ-LA-semiring and e be the identity of S. Let $x, y \in S$ and $\gamma, \beta \in \Gamma$. Then, we have

$$\begin{aligned} x\gamma y &= x\gamma(e\beta y) \\ &= e\gamma(x\beta y) \\ &= x\beta y \end{aligned}$$

Lemma 6. *Let S be a gamma LA semiring with identity and $x \in S$. If X is a left ideal of S then $X\gamma x$ is a left ideal in$S$, where$\gamma \in \Gamma$.*

Proof. If S is gamma LA-semiring having left identity and let $x \in S$. Now consider

$$\begin{aligned} s\gamma x + r\gamma x &= (s + r)\gamma x \\ &\in X\gamma x. \end{aligned}$$

And

$$S\Gamma(X\gamma x) \subseteq (S\Gamma X)\gamma x \subseteq X\gamma x$$

for all $r, s \in X$ and $\gamma \in \Gamma$. Hence $X\gamma x$ is a left ideal in$S$.

Corollary 2. *Let S be a gamma LA-semiring with identity and $x \in S$. If X is a right ideal of S, then $x\gamma X$ is a right ideal in S, where $\gamma \in \Gamma$.*

Proof. It is similar to the proof of Lemma 6.

Lemma 7. *Let S be a gamma LA-semiring with identity and X, Y be the left ideals of S. then, for each left ideal Y of S, $(X : \Gamma : Y)$ is a left ideal in S, where $(X : \Gamma : Y) = \{x \in S : x\Gamma Y \subseteq X\}$.*

Proof. Suppose that S is a gamma LA-semiring with left identity. Let $s \in S$ and let $x, y \in (X : \mathit{\Gamma} : Y)$. Then $x\mathit{\Gamma}Y \subseteq X$ and $y\mathit{\Gamma}Y \subseteq X$ so that

$$\begin{aligned}(x+y)\mathit{\Gamma}Y &= x\mathit{\Gamma}Y + y\mathit{\Gamma}Y \\ &\subseteq X + X \\ &= X.\end{aligned}$$

And

$$(s\gamma x)\mathit{\Gamma}Y = s\gamma(x\mathit{\Gamma}Y) \subseteq s\gamma X \ \subseteq X$$

for all $\gamma \in \mathit{\Gamma}$. Hence $x + y \in (X : \mathit{\Gamma} : Y)$ and $S\mathit{\Gamma}(X : \mathit{\Gamma} : Y) \subseteq (X : \mathit{\Gamma} : Y)$. Thus $(X : \mathit{\Gamma} : Y)$ is a left ideal in S.

Corollary 3. *Let S be a gamma LA-semiring with identity and X be a left ideal of S. Then, $(X : \gamma : r)$ is a left ideal in S, where $(X : \gamma : r) = \{x \in S : x\gamma r \in X\}$.*

Proof. This follows from lemma 7

Remark 4. *Let X, Y and Z be the left ideals of a gamma LA-semiring S. Then $(X : \mathit{\Gamma} : Z) \subseteq (X : \mathit{\Gamma} : Y)$, where $Y \subseteq Z$.*

Theorem 9. *Let S be a Γ-LA-semiring with identity. Then, $(X : \mathit{\Gamma} : Y)$ is a quasi-ideal in S if X is quasi-ideal of S.*

Proof. Assume that X is a quasi-ideal of S, then By Lemma 7, we have $(X : \mathit{\Gamma} : Y)$ is a left ideal in S. Then,

$$\begin{aligned}(S\mathit{\Gamma}\,(X : \mathit{\Gamma} : Y)) \cap ((X : \mathit{\Gamma} : Y)\mathit{\Gamma}S) &\subseteq (X : \mathit{\Gamma} : Y) \cap (X : \mathit{\Gamma} : Y) \\ &\subseteq (X : \mathit{\Gamma} : Y).\end{aligned}$$

Hence $(X : \mathit{\Gamma} : Y)$ is a quasi-ideal in S.

Theorem 10. *Let S be a gamma LA-semiring with identity. Then $(X : \mathit{\Gamma} : Y)$ is a left k-ideal in S, if X be a left k-ideal of S.*

Proof. Assume that X is a left k-ideal of S then By Lemma 7, $(X : \mathit{\Gamma} : Y)$ is a left ideal in S. Similarly, if $x, x + t \in (X : \mathit{\Gamma} : Y)$ then $x\mathit{\Gamma}Y \subseteq X$ and $(x + t)\mathit{\Gamma}Y \subseteq X$ that is $x\mathit{\Gamma}Y \subseteq X$ and $x\mathit{\Gamma}Y + t\mathit{\Gamma}Y \subseteq X$. Then, we get $t\mathit{\Gamma}Y \subseteq X$. Hence $(X : \mathit{\Gamma} : Y)$ is a left k-ideal in S.

3.2. ALMOST PRIME IDEALS IN Γ-LA-SEMIRING

In this section, we initiate the term almost prime and weakly almost prime ideals in Γ-LA- semiring. Our starting point is the following definition.

Definition 10. *A left ideal P is called almost-prime if $X\Gamma Y \subseteq P$ implies that $X \subseteq P$ or $Y \subseteq P$, where X and Y are respectively left and right ideal of S.*

Remark 5. *It is easy to see that every almost-prime left ideal is prime.*

Definition 11. *A left ideal P is called weakly almost-prime if $\{0\} \neq X\Gamma Y \subseteq P$ implies $X \subseteq P$ or $Y \subseteq P$, where X and Y are respectively left and right ideal of S.*

Remark 6. *It is easy to see that every almost-prime left ideal is weakly almost-prime.*

Lemma 8. *Let P be the ideal of a Γ-LA-semiring S with identity. Then P is an almost-prime left ideal of S if $x\Gamma(S\Gamma y) \subseteq P$ implies $x \in P$ or $y \in P$.*

Proof. Let P be an almost-prime left ideal of a Γ-LA-semiring S with identity. Now suppose that $x\Gamma(S\Gamma y) \subseteq P$. Then by hypothesis, we have

$$\begin{aligned}(S\Gamma x)\Gamma(y\Gamma S) &\subseteq (S\Gamma x)\Gamma S\Gamma(y\Gamma S)\\ &= (x\Gamma S)\Gamma S\Gamma(S\Gamma y)\\ &= (S\Gamma S)\Gamma x\Gamma(S\Gamma y)\\ &\subseteq (S\Gamma S)\Gamma P\\ &= (P\Gamma S)\Gamma S\\ &\subseteq P\Gamma S \subseteq P\end{aligned}$$

which implies $(S\Gamma x)\Gamma(y\Gamma S) \subseteq P$. Then, $x = e\gamma x \in S\Gamma x \subseteq P$ or $y = y\gamma e \in y\Gamma S \subseteq P$. Hence $x \in P$ or$y \in P$.

Corollary 4. *Let P be an almost-prime left ideal of a Γ-LA-semiring S with identity. Then P is a weakly almost-prime left ideal of S if $\{0\} \neq x\Gamma(S\Gamma y) \subseteq P$, then $x \in P$ or $y \in P$.*

Proof. This follows from Lemma 8

Theorem 11. *Let S be a gamma LA-semiring with identity and $x, y \in S$ and $\gamma \in \Gamma$. Then a left ideal P of S is almost-prime iff $x\gamma y \in P$ implies $x \in P$ or$y \in P$.*

Proof. Let P be a left ideal of a Γ-LA-semiring with identity. Now suppose that $x\gamma y \in P$, where $x, y \in S$ and $\gamma \in \Gamma$. Then by hypothesis, we get;

$$(S\Gamma x)\gamma(y\Gamma S) \subseteq S\Gamma((x\gamma y)\Gamma S)$$

$$\subseteq S\Gamma(P\Gamma S)$$
$$\subseteq S\Gamma P$$
$$\subseteq P.$$

So by the definition of almost-prime, we have $x \in P$ or $y \in P$.
Conversely, assume that if $x\gamma y \in P$ implies $x \in P$ or $y \in P$ and X is left ideal of S. Let $X\Gamma Y \subseteq P$, where Y is right ideal of S such that $Y \subseteq S - P$. Then there exists $y \in Y$ such that $y \notin P$. Now we get $x\gamma y \in P$. So by hypothesis, $x \in P$, for all $x \in X \Longrightarrow X \subseteq P$. So P is almost-prime left ideal in S.

Corollary 5. *Let S be a gamma LA-semiring having identity and let $x, y \in S$, $\gamma \in \Gamma$. Then a left ideal P of S is weakly almost-prime iff $0 \neq x\gamma y \in P$ implies $x \in P$ or $y \in P$.*

Proof. This follows from Theorem 11

Theorem 12. *Let S be a gamma LA-semiring having left identity and X be an almost-prime left ideal of S. Then $(X : \Gamma : Y)$ is an almost-prime left ideal in S, where $Y \subseteq S-X$.*

Proof. Assume that X is a almost-prime left ideal of S. By Lemma 7, we have $(X : \Gamma : Y)$, a left ideal in S. Let $x\gamma y \in (X : \Gamma : Y)$, where $x, y \in S$ and $\gamma \in \Gamma$. Suppose that $y \notin (X : \Gamma : Y)$. Since $x\gamma y \in (X : \Gamma : Y)$, we have $(x\gamma y)\Gamma Y \subseteq X$. So by hypothesis

$$(S\Gamma x)\gamma(y\Gamma Y) = S\Gamma((x\gamma y)\Gamma Y)$$
$$\subseteq S\Gamma X$$
$$\subseteq X.$$

Then, following the definition of almost-prime, we have $x = e\gamma x \in S\Gamma x \subseteq X$ or $y\Gamma Y \subseteq X$ implies that $x\Gamma S \subseteq X\Gamma S \subseteq X$. Hence $(X : \Gamma : Y)$ is an almost-prime left ideal in S.

Corollary 6. *Let S be a gamma LA-semiring having left identity and let X be an ideal of S. If X is a weakly almost-prime left ideal of S, then $(X : \Gamma : Y)$, is a weakly almost-prime left ideal in S, where $Y \subseteq S - X$.*

Proof. This follows from Theorem 12

Corollary 7. *Let S be a gamma LA-semiring having left identity and let X be an ideal of S. If X is an almost-prime left ideal of S, then $(X : \gamma : s)$ is an almost-prime left ideal in S, where $s \in S - X$ and $\gamma \in \Gamma$.*

Proof. This follows from Theorem 12

Corollary 8. *Let S be a gamma LA-semiring with left identity and let X be an ideal of S. If X is a weakly almost-prime left ideal of S, then $(X : \gamma : s)$, is a weakly almost-prime left ideal in S, where $s \in S - X$ and $\gamma \in \Gamma$.*

Proof. This follows from Corollary 5

4. Conclusion

In this manuscript, firstly we have added some fresh examples along with new results in Γ-LA-rings. Next, we introduced the notion of Γ-LA-semirings and discussed different types of ideals in Γ-LA-semirings. We examined that almost all the results of LA-rings and LA-semirings are valid in case of Γ-LA-rings and Γ-LA-semirings. One could extend this work by shifting our results towards the theory of Γ-LA-nearrings, Γ-LA-hemirings etc.

Acknowledgements

Some results presented in this article are extracted from MS thesis of fourth author (Zahid Hussain) which was written under the supervision of the first author (Waheed Ahmad Khan) and was submitted to Higher Education of Pakistan.

References

[1] W. Barnes. On the gamma-rings of nobusawa. *Pacific Journal of Mathematics*, 18(3):411–422, 1966.

[2] G. Booth and N. Groenewald. Equiprime gamma-near-rings. *Quaestiones Mathematicae*, 14:411–417, 1991.

[3] J. R. Cho, J. Jezek Pusan, and T. Kepka. Praha, paramedial groupoids. *Czechoslovak Mathematical Journal*, 49:124, 1996.

[4] D. M. Devi and G. S. Latha. La–semirings satisfying the identity ab= a+ b+ 1. *International Journal of Innovative Science, Engineering & Technology*, 2:378–389, 2015.

[5] T. K. Dutta and S. K. Sardar. On prime ideals and prime radicals of a gamma-semirings. *An. Stiint. Univ. Al. I. Cuza Iasi, Mat.(NS)*, 46:319–329, 2000.

[6] T. K. Dutta and S. K. Sardar. Semiprime ideals and irreducible ideals of r-semirings. *Novi Sad J. Math*, 30(1):97–108, 2000.

[7] R. D. Jagatap and Y. S. Pawar. Quasi-ideals and minimal quasi-ideals in gamma-semirings. *Novi Sad J. Math*, 39(2):79–87, 2009.

[8] R. D. Jagatap and Y. S. Pawar. Quasi-ideals in regular gamma-semirings. *Bulletin of Kerala Mathematics Association*, 7(2):51–61, 2010.

[9] Waheed Ahmad Khan, Abdelghani Taouti, Seema Karkain, Azar Salami, and Waqar Arif. Weakly prime and weakly primary ideals in gamma seminearrings. *European Journal of Pure and Applied Mathematics*, 12(2):544–552, 2019.

[10] Q. Mushtaq and M. Khan. Ideals in la-semigroups. In *Proc. of 4th International Pure Mathematics Conference*, 2003.

[11] Q. Mushtaq and S. M. Yusuf. On la-semi groups. *the Alig. Bull.Math.*, 8:65–70, 1978.

[12] Q. Mushtaq and S. M. Yusuf. On locally associative la-semigroups. *J. Nat. Sci. Math*, 19(1):57–62, 1979.

[13] N. Nobusawa. On a generalization of the ring theory. *Osaka Journal of Mathematics*, 1(1):81–89, 1964.

[14] M. M. K. Rao. Gamma-semirings-i. *Southeast Asian Bull. Math*, 19(1):49–54, 1995.

[15] M. M. K. Rao. Ideals in ordered gamma-semirings. *Discussiones Mathematicae-General Algebra and Applications*, 38(1):47–68, 2018.

[16] M. Sarwar. *Conditions for LA-semigroups to resemble associative structures.* PhD thesis, Quaid-i-Azam University Islamabad, 1993.

[17] B. Satyanarayana. A note on gamma-near-rings. *Indian J. Math*, 41(3):427–433, 1999.

[18] T Shah and I Rehman. On la-rings of finitely non-zero functions. *Int. J. Contemp. Math. Sciences*, 5(5):209–222, 2010.

[19] P. Yiarayong. On left primary and weakly left primary ideals in gamma-la-rings. *Gazi University Journal of Science*, 29(1):143–148, 2016.

[20] S. M. Yusuf. On left almost ring. In *Proc. of 7th International Pure Math. Conference, Islamabad*, 2006.