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A note on the definite integral of the Lerch function

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Abstract. In this manuscript, the authors derive a formula for the double Laplace transform expressed in terms of the Lerch Transcendent. The log term mixes the variables so that the integral is not separable except for special values of k. The method of proof follows the method used by us to evaluate single integrals. This transform is then used to derive definite integrals in terms of fundamental constants, elementary and special functions. A summary of the results is produced in the form of a table of definite integrals for easy referencing by readers. The majority of the results in the work are new.

2020 Mathematics Subject Classifications: 01A35, 11M06, 11M35, 30-02, 30D10, 30D30, 30E20

Key Words and Phrases: Laplace transform, Lerch Transcendent, Contour integral, definite integral

1. Significance Statement

Definite integrals of special functions occur in a wide range of applications. Such applications of these integrals are prominent in diffusion theory [8], transportation problems [8], the study of the radiative equilibrium of stellar atmospheres [6] and the evaluation of exchange integrals in quantum mechanics [7]. This paper is an effort to give a tabulation of an integral for a particular special function not present in current literature. The special function researched in this work is the Lerch function.

In 1887 Mathias Lerch [9] produced his famous manuscript on the Lerch function. Lerch's function has been extensively in studied in [3–5, 9, 10, 13]. The Lerch function generalizes the Hurwitz zeta function, the polylogarithms, and many interesting and important special functions. Definite integrals of special functions such as Hurwitz zeta and Polylogarithm have been studied in the works of Kurokawa et al and Reynolds and Stauffer [8, 14]. Relations between the Hurwitz-Lerch zeta functions and Appel functions as

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well as Humbert hypergeometric functions of two variables are found in the work of [2].

In this work our goal is to expand upon the current literature of definite integrals of special functions by providing a formal derivation of the definite integral of the Lerch function and express this integral in terms of the Lerch function. It is our hope that researches will find this new integral formula useful for current and future research work where applicable. Consequently, any new result on the Lerch function is important because of its many applications in applied and pure mathematics.

2. Introduction

In the present work, the authors used their contour integral method and applied it to the Lerch function to derive a definite integral and expressed its closed form in terms of a special function. This derived integral formula was then used to provide formal derivations in terms of special functions and fundamental constants. The Lerch function being a special function has the fundamental property of analytic continuation, which enables us to widen the range of evaluation for the parameters involved in our definite integral. The Lerch function is a special function that generalizes the Hurwitz zeta function, the polylogarithms, and so many interesting and important special functions.

The definite integral derived in this manuscript is given by

$$\int_0^{+\infty} x^{a-1}\log^k\left(\frac{b}{x}\right)\Phi(-cx,n,a)dx = \frac{e^{i\pi a}k!c^{-a}(-2i\pi)^{k+n+1}\Phi\left(e^{2ia\pi},-k-n,-\frac{-i\log(b)-i\log(c)-\pi}{2\pi}\right)}{(k+n)!} \tag{1}$$

where the parameters k, b and a are general complex numbers and $0 < Re(a) < 1, c \in \mathbb{R}_+, n \in \mathbb{Z}_-$. This work is important because the authors were unable to find similar derivations in the current literature along with the many applications the Lerch function has in applied and pure mathematics. The derivation of the definite integral follows the method used by us in [15] which involves Cauchy's integral formula. The generalized Cauchy's integral formula is given by

$$\frac{y^k}{k!} = \frac{1}{2\pi i}\int_C \frac{e^{wy}}{w^{k+1}}dw. \tag{2}$$

where C is in general an open contour in the complex plane where the bilinear concomitant [15]. This method involves using a form of equation (2) then multiply both sides by a function, then take a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying equation (2) by a function and performing some substitutions so that the contour integrals are the same.

3. Definite integral of the contour integral

We use the method in [15]. The variable of integration in the contour integral is $t = w - a$. The cut and contour are in the second quadrant of the complex z-plane. The cut approaches the origin from the interior of the second quadrant and the contour goes round the origin with zero radius and is on opposite sides of the cut. Using equation (2) we replace y by $1/x + \log(b)$ then multiply by $x^{a-1}\Phi(-cx, n, a)$. Next we take the infinite integral over $x \in [0, +\infty)$ to get

$$
\begin{aligned}
&\frac{1}{k!}\int_0^{+\infty} x^{a-1}\log^k\left(\frac{b}{x}\right)\Phi(-cx, n, a)dx \\
&\qquad = \frac{1}{2\pi i}\int_0^{+\infty}\int_C b^w w^{-k-1} x^{a-w-1}\Phi(-cx, n, a)dwdx \\
&\qquad = \frac{1}{2\pi i}\int_C\int_0^{+\infty} b^w w^{-k-1} x^{a-w-1}\Phi(-cx, n, a)dxdw \\
&\qquad = \frac{1}{2\pi i}\int_C \pi b^w c^{w-a}\csc(\pi(a-w))w^{-k-n-1}dw
\end{aligned} \tag{3}
$$

where $-1 < Re(a - w) < 0$. Using equation (9.550) we multiply both sides by x^{b-1} and integrate over $x \in [0, +\infty)$ and using equation (3.194.4) in [17]. We are able to switch the order of integration over z, x and y using Fubini's theorem since the integrand is of bounded measure over the space $C \times [0, +\infty)$.

4. The Lerch function

We use (9.550) and (9.556) in [17] where $\Phi(z, s, v)$ is the Lerch function which is a generalization of the Hurwitz zeta $\zeta(s, v)$ and Polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$
\Phi(z, s, v) = \sum_{n=0}^{+\infty}(v + n)^{-s} z^n \tag{4}
$$

where $|z| < 1, v \neq 0, -1, ..$ and is continued analytically by its integral representation given by

$$
\Phi(z, s, v) = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}e^{-vt}}{1 - ze^{-t}}dt = \frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{t^{s-1}e^{-(v-1)t}}{e^t - z}dt \tag{5}
$$

where $Re(v) > 0$, or $|z| \leq 1, z \neq 1, Re(s) > 0$, or $z = 1, Re(s) > 1$.

5. Infinite sum of the contour integral

In this section we will again use Cauchy's integral formula (2) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed using equation (2) and replace y by $\log(b)+\log(c)+i\pi t(2y+1)$ and multiply both sides by $-2i\pi c^{-a}e^{2i\pi ay+i\pi a}$ and set $t=-1$ and replace k by $k+n$ simplifying to get

$$-\frac{2i\pi c^{-a}e^{i\pi a(2y+1)}(-i)^{k+n}(i\log(b)+i\log(c)+2\pi y+\pi)^{k+n}}{(k+n)!} = -\frac{1}{2\pi i}\int_C 2i\pi b^w c^{w-a}e^{i\pi(2y+1)(a-w)}w^{-k-n-1}dw \quad (6)$$

Next take the infinite over $y\in[0,+\infty)$ and simplify using the Lerch function to get

$$\begin{aligned}\frac{e^{i\pi a}c^{-a}(-2i\pi)^{k+n+1}\Phi\left(e^{2ia\pi},-k-n,-\frac{-i\log(b)-i\log(c)-\pi}{2\pi}\right)}{(k+n)!} &= -\frac{1}{2\pi i}\sum_{y=0}^{+\infty}\int_C 2i\pi b^w c^{w-a}e^{i\pi(2y+1)(a-w)}w^{-k-n-1}dw \\ &= -\frac{1}{2\pi i}\int_C\sum_{y=0}^{+\infty}2i\pi b^w c^{w-a}e^{i\pi(2y+1)(a-w)}w^{-k-n-1}dw \\ &= \frac{1}{2\pi i}\int_C \pi b^w c^{w-a}\csc(\pi(a-w))w^{-k-n-1}dw \quad (7)\end{aligned}$$

from (1.232.3) in [17] and $Im(a-w)>0$ for convergence of the sum.

6. Definite integral in terms of the Lerch function

In this section we derive the definite integral involving the Lerch and logarithmic functions expressed in terms of the Lerch function.

Theorem 1. *For $k,b\in\mathbb{C},0<Re(a)<1,n\in\mathbb{Z}_-,c\in\mathbb{Z}$,*

$$\int_0^{+\infty}x^{a-1}\log^k\left(\frac{b}{x}\right)\Phi(-cx,n,a)dx = \frac{e^{i\pi a}k!c^{-a}(-2i\pi)^{k+n+1}\Phi\left(e^{2ia\pi},-k-n,-\frac{-i\log(b)-i\log(c)-\pi}{2\pi}\right)}{(k+n)!} \quad (8)$$

Proof. Since the right-hand sides of equations (3) and (7) are the same we can equate the left-hand sides to yield the stated result.

Theorem 2. $b\in\mathbb{C},Re(k)<-2,0<Re(a)<1,c\in\mathbb{Z}_-$

$$\int_0^{+\infty}x^{a-1}\Phi(-cx,1,a)\log^k\left(\frac{b}{x}\right)dx$$

$$= \frac{e^{i\pi a}(-2i\pi)^{k+2}k!c^{-a}\Phi\left(e^{2ia\pi}, -k-1, -\frac{-i\log(b)-i\log(c)-\pi}{2\pi}\right)}{(k+1)!} \quad (9)$$

Proof. Use equation (8) set $n = -1$ and simplify.

7. Definite integrals in terms of the Hurwitz zeta function when $a = 1/2$ and $a = 1$

Theorem 3. *For $k, b, c \in \mathbb{C}, n \in \mathbb{Z}_-$*

$$\int_0^{+\infty} \frac{\log^k\left(\frac{b}{x}\right) Li_n(-cx)}{x} dx = \frac{k!(-2i\pi)^{k+n+1}\zeta\left(-k-n, -\frac{-i\log(b)-i\log(c)-\pi}{2\pi}\right)}{(k+n)!} \quad (10)$$

Proof. Use equation (8) and set $a = 1$ and simplify using equations (64:12:1) and (64:12:2) in [11].

Proposition 1. *For $b, c \in \mathbb{C}, Re(k) < 0$*

$$\int_0^{+\infty} \frac{\tanh^{-1}(cx)\log^k\left(\frac{b}{x}\right)}{x} dx$$

$$= \frac{(-i)^k 2^{k+1}\pi^{k+2}\left(\zeta\left(-k-1, \frac{i\log(b)+i\log(c)+\pi}{2\pi}\right) - \zeta\left(-k-1, \frac{i\log(b)+i\log(-c)+\pi}{2\pi}\right)\right)}{k+1} \quad (11)$$

Proof. Use equation (10) set $n = 1$, simplify the factorial and form a second equation by replacing c by $-c$ and taking their difference and simplify.

Proposition 2. *For $\alpha, \beta \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{\tanh^{-1}(\alpha x)}{x(\beta + \log(x))^2} dx$$

$$= \frac{1}{2}\left(\psi^{(0)}\left(-\frac{i(\beta - \log(\alpha))}{2\pi}\right) - \psi^{(0)}\left(\frac{-i\beta + i\log(\alpha) + \pi}{2\pi}\right)\right) \quad (12)$$

Proof. Use equation (11) set $k = -2, b = e^{-i\beta}, c = \alpha$ and simplify using equation (64:4:1) in [11]. Note the singularity at $x = 1/\alpha$.

Proposition 3.

$$\int_0^{+\infty} \frac{\left(\log^2(x) - \pi^2\right)\tanh^{-1}(x)}{x\left(\log^2(x) + \pi^2\right)^2} dx = \frac{1}{2}\left(\gamma + \psi^{(0)}\left(\frac{1}{2}\right)\right) \quad (13)$$

Proof. Use equation (12) and set $\beta = \pi i, \alpha = 1$ and simplify in terms of Euler's constant, γ using equation (6.3.16) in [1].

Theorem 4. *For $k, b, c \in \mathbb{C}$*

$$\begin{aligned}\int_0^{+\infty} & \frac{\log^k\left(\frac{b}{x}\right)\left(\tan^{-1}\left(\sqrt{d}\sqrt{x}\right)-\tan^{-1}\left(\sqrt{c}\sqrt{x}\right)\right)}{x}dx \\ &= -\frac{(-4i)^{k+1}\pi^{k+2}\zeta\left(-k-1, \frac{i\log(b)+i\log(c)+\pi}{4\pi}\right)}{k+1} \\ &+ \frac{(-4i)^{k+1}\pi^{k+2}\zeta\left(-k-1, \frac{i\log(b)+i\log(c)+3\pi}{4\pi}\right)}{k+1} \\ &+ \frac{(-4i)^{k+1}\pi^{k+2}\zeta\left(-k-1, \frac{i\log(b)+i\log(d)+\pi}{4\pi}\right)}{k+1} \\ &- \frac{(-4i)^{k+1}\pi^{k+2}\zeta\left(-k-1, \frac{i\log(b)+i\log(d)+3\pi}{4\pi}\right)}{k+1} \end{aligned} \tag{14}$$

Proof. Use equation (8) and set $a = 1/2, n = -1$ then take the first partial derivative with respect to k then set $k = 1$ and simplify using entry (4) in Table below (64:12:7) in [11].

Proposition 4. *For $d, c \in \mathbb{C}$,*

$$\int_0^{+\infty} \frac{\tan^{-1}(dx)-\tan^{-1}(cx)}{x}dx = \frac{1}{2}\pi\log\left(\frac{d}{c}\right) \tag{15}$$

Proof. Use equation (9) set $a = 1/2$ and simplify using entry (4) in table below (64:12:7) in [11]. Next form a second equation by replacing c by d and take their difference, and setting $k = 0$ and simplify the integral where d is replaced by d^2, c is replaced by c^2 and replacing x by x^2 where $dx = 2xdx$. This is a particular case of the well-known Frullani integrals in section (2.5) in [16].

Proposition 5. *For $b, d, c \in \mathbb{C}$,*

$$\int_0^{+\infty} \frac{\log\left(\frac{b}{x}\right)\left(\tan^{-1}\left(\sqrt{d}\sqrt{x}\right)-\tan^{-1}\left(\sqrt{c}\sqrt{x}\right)\right)}{x}dx = -\frac{1}{4}\pi\log\left(\frac{c}{d}\right)\log\left(b^2cd\right) \tag{16}$$

Proof. Use equation (9) set $a = 1/2$ and simplify using entry (4) in table below (64:12:7) in [11]. Next form a second equation by replacing c by d and take their difference, and setting $k = 1$ and simplify the integral where d is replaced by d^2, c is replaced by c^2 and replacing x by x^2 where $dx = 2xdx$.

Proposition 6. *For $b, d, c \in \mathbb{C}$,*

$$\int_0^{+\infty} \frac{\log^2\left(\frac{b}{x}\right)\left(\tan^{-1}\left(\sqrt{d}\sqrt{x}\right) - \tan^{-1}\left(\sqrt{c}\sqrt{x}\right)\right)}{x} dx$$
$$= -\frac{1}{2}\pi\log(b)\log^2(c) - \frac{1}{2}\pi\log^2(b)\log(c) + \frac{1}{2}\pi\log(b)\log^2(d)$$
$$+ \frac{1}{2}\pi\log^2(b)\log(d) - \frac{1}{6}\pi\log^3(c) - \frac{1}{2}\pi^3\log(c) + \frac{1}{6}\pi\log^3(d)$$
$$+ \frac{1}{2}\pi^3\log(d) \quad (17)$$

Proof. Use equation (9) set $a = 1/2$ and simplify using entry (4) in table below (64:12:7) in [11]. Next form a second equation by replacing c by d and take their difference, and setting $k = 2$ and simplify the integral by setting $d = d^2, c = c^2$ and replacing x by x^2 where $dx = 2xdx$.

Proposition 7. *Using equation (14) and setting $k = 1/2, b = -i, c = 1, d = -1$ simplifying to get*

$$\int_0^{+\infty} -\frac{\sqrt{\log\left(-\frac{i}{x}\right)}\left(\tan^{-1}\left(\sqrt{x}\right) - i\tanh^{-1}\left(\sqrt{x}\right)\right)}{x} dx$$
$$= -\frac{16}{3}\sqrt[4]{-1}\pi^{5/2}\left(\zeta\left(-\frac{3}{2},\frac{1}{8}\right) - \zeta\left(-\frac{3}{2},\frac{3}{8}\right) - \zeta\left(-\frac{3}{2},\frac{5}{8}\right) + \zeta\left(-\frac{3}{2},\frac{7}{8}\right)\right) \quad (18)$$

Theorem 5. *For $k, b, c \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{(cx-1)\log^k\left(\frac{b}{x}\right)}{\sqrt{x}(cx+1)^2} dx$$
$$= \frac{ie^{-\frac{1}{2}i\pi k}k(4\pi)^k\left(\zeta\left(1-k, \frac{i\log(b)+i\log(c)+3\pi}{4\pi}\right) - \zeta\left(1-k, \frac{i\log(b)+i\log(c)+\pi}{4\pi}\right)\right)}{\sqrt{c}} \quad (19)$$

Proof. Use equation (8) and set $a = 1/2, n = -1$ then take the first partial derivative with respect to k then set $k = 1$ and simplify using entry (4) in Table below (64:12:7) in [11].

Proposition 8. *Using equation (19) setting $k = -1, b = 1, c = 1$ and simplifying in terms of Catalan's constant, C using equations (25.11.35) and (25.11.40) in [12] to get*

$$\int_0^{+\infty} \frac{1-x}{\sqrt{x}(x+1)^2\log(x)} dx = -\frac{4C}{\pi} \quad (20)$$

Proposition 9. *Using equation (19) setting $k = -1/2, b = -i, c = 1$ and simplifying to get*

$$\int_0^{+\infty} \frac{x-1}{\sqrt{x}(x+1)^2\sqrt{\log\left(-\frac{i}{x}\right)}} dx = \frac{(-1)^{3/4}\left(\zeta\left(\frac{3}{2},\frac{3}{8}\right) - \zeta\left(\frac{3}{2},\frac{7}{8}\right)\right)}{4\sqrt{\pi}} \quad (21)$$

Proposition 10. *Using equation (19) setting $k = 1/2, b = -i, c = 1$ and simplifying to get*

$$\int_0^{+\infty} \frac{(x-1)\sqrt{\log\left(-\frac{i}{x}\right)}}{\sqrt{x}(x+1)^2} dx = \sqrt[4]{-1}\sqrt{\pi}\left(\zeta\left(\frac{1}{2},\frac{7}{8}\right) - \zeta\left(\frac{1}{2},\frac{3}{8}\right)\right) \tag{22}$$

8. Definite integrals in terms of the Lerch function when $n \in \mathbb{Z}_-$

Proposition 11. *For $k, b \in \mathbb{C}$*

$$\begin{aligned}&\int_0^{+\infty} \frac{((x-6)x+1)\log^k\left(\frac{b}{x}\right)}{4\sqrt{x}(x+1)^3} dx \\ &= 2^{2k-3} e^{-\frac{1}{2}i\pi k}(k-1)k\pi^{k-1}\left(\zeta\left(2-k, \frac{i\log(b)}{4\pi} + \frac{3}{4}\right) - \zeta\left(2-k, \frac{i\log(b)+\pi}{4\pi}\right)\right)\end{aligned} \tag{23}$$

Proof. Use equation (8) and set $a = 1/2, n = -2, c = 1$ then simplify using entry (4) in Table below (64:12:7) in [11].

Proposition 12. *Using equation (23) and setting $k = -1, b = -i$ and simplifying we get*

$$\int_0^{+\infty} \frac{(x-6)x+1}{\sqrt{x}(x+1)^3\left(4\log^2(x)+\pi^2\right)} dx = \frac{\zeta\left(3,\frac{7}{8}\right) - \zeta\left(3,\frac{3}{8}\right)}{8\pi^3} \tag{24}$$

Theorem 6.

$$\int_0^{+\infty} \frac{((x-6)x+1)\log\left(-\log(x)+\frac{i\pi}{2}\right)}{\sqrt{x}(x+1)^3} dx = \frac{\psi^{(1)}\left(\frac{3}{8}\right) - \psi^{(1)}\left(\frac{7}{8}\right)}{2\pi} \tag{25}$$

Proof. Use equation (23) take the first partial derivative with respect to k then set $k = 0, b = -i$ and simplify.

Proposition 13. *For $k, b, c \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{(cx-1)\log^k\left(\frac{b}{x}\right)}{(cx+1)^3} dx = \frac{ie^{-\frac{1}{2}i\pi k}(k-1)k(2\pi)^{k-1}\zeta\left(2-k, \frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)}{c} \tag{26}$$

Proof. Use equation (8) and set $a = 1, n = -2$ then simplify using entry (4) in Table below (64:12:7) in [11].

Proposition 14. *Using equation (26) and setting $k = -1, b = -i, c = 1$, rationalizing the denominator and simplify we get*

$$\int_0^{+\infty} \frac{(x-1)\log(x)}{(x+1)^3\left(4\log^2(x)+\pi^2\right)} dx = \frac{\zeta\left(3,\frac{3}{4}\right)}{8\pi^2} \tag{27}$$

and

$$\int_0^{+\infty} \frac{x-1}{(x+1)^3\left(4\log^2(x)+\pi^2\right)} dx = 0 \tag{28}$$

Theorem 7. *For $k, b, c \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{(cx(cx-4)+1)\log\left(\log\left(\frac{b}{x}\right)\right)\log^k\left(\frac{b}{x}\right)}{(cx+1)^4}dx$$
$$= -\frac{i2^{k-3}e^{-\frac{1}{2}ik\pi}\pi^{k-1}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^3}{c}$$
$$-\frac{e^{-\frac{1}{2}ik\pi}(2\pi)^{k-2}\zeta'\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^3}{c}$$
$$+\frac{e^{-\frac{1}{2}ik\pi}(2\pi)^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)\log(2\pi)k^3}{c}$$
$$+\frac{3e^{-\frac{1}{2}ik\pi}(2\pi)^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^2}{c}$$
$$+\frac{3i2^{k-3}e^{-\frac{1}{2}ik\pi}\pi^{k-1}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^2}{c}$$
$$+\frac{2^{k-1}e^{-\frac{1}{2}ik\pi}\pi^{k-2}\zeta'\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^2}{c}$$
$$+\frac{e^{-\frac{1}{2}ik\pi}(2\pi)^{k-2}\zeta'\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k^2}{c}$$
$$-\frac{3e^{-\frac{1}{2}ik\pi}(2\pi)^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)\log(2\pi)k^2}{c}$$
$$-\frac{3\ 2^{k-1}e^{-\frac{1}{2}ik\pi}\pi^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k}{c}$$
$$-\frac{i2^{k-2}e^{-\frac{1}{2}ik\pi}\pi^{k-1}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k}{c}$$
$$-\frac{2^{k-1}e^{-\frac{1}{2}ik\pi}\pi^{k-2}\zeta'\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)k}{c}$$
$$+\frac{2^{k-1}e^{-\frac{1}{2}ik\pi}\pi^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)\log(2\pi)k}{c}$$
$$+\frac{2^{k-1}e^{-\frac{1}{2}ik\pi}\pi^{k-2}\zeta\left(3-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)}{c} \quad (29)$$

Proof. Use equation (8) and set $a=1, n=-3$ then simplify using entry (4) in Table below (64:12:7) in [11]. Then we take the first partial derivative with respect to k and set $k=0, b=1, c=1$ and simplify.

Proposition 15. *Using equation (29) and setting $k=-1, b=-i, c=1$ and simplifying we get*

$$\int_0^{+\infty} \frac{((x-4)x+1)\log\left(\log\left(\frac{1}{x}\right)\right)}{(x+1)^4} dx = \frac{7\zeta(3)}{2\pi^2} \tag{30}$$

Note there exists a singularity at $x=1$.

Proposition 16. *For $k, b, c \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{(cx-1)(cx(cx-10)+1)\log^k\left(\frac{b}{x}\right)}{(cx+1)^5} dx$$
$$= -\frac{ie^{-\frac{1}{2}i\pi k}(k-3)(k-2)(k-1)k(2\pi)^{k-3}\Phi\left(1, 4-k, \frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)}{c} \tag{31}$$

Proof. Use equation (8) and set $a=1, n=-4$ then simplify using entry (4) in Table below (64:12:7) in [11].

Theorem 8.

$$\int_0^{+\infty} \frac{(x-1)((x-10)x+1)\log\left(\frac{1}{x}\right)\log\left(\log\left(\frac{1}{x}\right)\right)}{(x+1)^5} dx = -\frac{7\zeta(3)}{2\pi^2} \tag{32}$$

Proof. Use equation (31) setting $a=1$ and simplify, next take the first partial derivative with respect to k and set $k=1, b=1, c=1$ and simplify using equation (64:12:1) and entry (2) in table below (64:7) in [11].

Proposition 17. *For $k \in \mathbb{C}$*

$$\int_0^{+\infty} \frac{(x-1)((x-22)x+1)\log^k\left(\frac{1}{x}\right)}{8\sqrt{x}(x+1)^4} dx$$
$$= i2^{2k-5}e^{-\frac{1}{2}i\pi k}(k-2)(k-1)k\pi^{k-2}\left(\zeta\left(3-k, \frac{1}{4}\right) - \zeta\left(3-k, \frac{3}{4}\right)\right) \tag{33}$$

Proof. Use equation (8) and set $a=1, n=-4$ then simplify using entry (4) in Table below (64:12:7) in [11].

Theorem 9.

$$\int_0^{+\infty} \frac{(x-1)((x-22)x+1)\log\left(\frac{1}{x}\right)\log\left(\log\left(\frac{1}{x}\right)\right)}{8\sqrt{x}(x+1)^4} dx = -\frac{2C}{\pi} \tag{34}$$

Proof. Use equation (33) and take the first partial derivative with respect to k and set $k=1, b=1$ and simplify using equation (64:7:1) in [11] and equation (16) in [2].

9. Definite integral of the Lerch function in terms of the Lerch transformation

In this section we will apply the Lerch transformation derived by Oberhettinger in [10] and apply it to our definite integral of the Lerch function and evaluate a few examples.

Theorem 10. *For $k, b, a \in \mathbb{C}, n \in \mathbb{Z}_-, c \in \mathbb{R}_+$,*

$$\begin{aligned}\frac{e^{-i\pi a}c^a(-2i\pi)^{-k-n-1}(k+n)!}{k!}\int_0^{+\infty} x^{a-1}\log^k\left(\frac{b}{x}\right)\Phi(-cx,n,a)dx \\ = ib^{a-1}c^{a-1}(2\pi)^{-k-n-1}e^{-\frac{1}{2}i\pi(2a+k+n)}\Gamma(k+n+1)\Phi\left(-\frac{1}{bc},k+n+1,1-a\right) \\ - ib^{a+1}c^{a+1}(2\pi)^{-k-n-1}e^{i\pi(k+n)-\frac{1}{2}i\pi(2a+k+n)}\Gamma(k+n+1)\Phi(-bc,k+n+1,a+1) \\ + b^a c^a(2\pi)^{-k-n-1}(-ia)^{-k-n-1}e^{\frac{1}{2}i\pi(k+n)-\frac{1}{2}i\pi(2a+k+n)}\Gamma(k+n+1)\end{aligned} \tag{35}$$

Proof. Use equation (12) in [10] and simplify.

Proposition 18. *Using equation (35) setting $n=0, c=1, b=-i, a=1/3, k=1/2$ and simplify to get*

$$\int_0^{+\infty}\frac{(-1)^{5/12}\sqrt{\log\left(-\frac{i}{x}\right)}}{\sqrt{2}\pi^{3/2}x^{2/3}(2x+2)}dx = \frac{\left(\frac{1}{8}+\frac{i}{8}\right)\left(\Phi\left(-i,\frac{3}{2},\frac{2}{3}\right)+i\Phi\left(i,\frac{3}{2},\frac{4}{3}\right)+3\sqrt{3}\right)}{\pi} \tag{36}$$

Proposition 19. *Using equation (35) setting $n=-1, c=1/2, b=-i, a=1/4, k=-1/2$ and simplify to get*

$$\begin{aligned}\int_0^{+\infty}\frac{\sqrt[4]{2}\sqrt{\pi}(2-3x)}{x^{3/4}(x+2)^2\sqrt{\log\left(-\frac{i}{x}\right)}}dx \\ = \sqrt{-1+i}\pi\left(\Phi\left(\frac{i}{2},-\frac{1}{2},\frac{5}{4}\right)-i\left(1+4\Phi\left(-2i,-\frac{1}{2},\frac{3}{4}\right)\right)\right)\end{aligned} \tag{37}$$

Theorem 11.

$$\begin{aligned}\int_0^{+\infty}\frac{(3x-2)\log\left(-\frac{i}{x}\right)\log\left(\log\left(-\frac{i}{x}\right)\right)}{x^{3/4}(x+2)^2}dx \\ = -(-1)^{3/8}\Phi'\left(\frac{i}{2},1,\frac{5}{4}\right)-4(-1)^{3/8}\Phi'\left(-2i,1,\frac{3}{4}\right) \\ +8(-1)^{7/8}\,{}_2F_1\left(\frac{1}{4},1;\frac{5}{4};\frac{i}{2}\right)-8(-1)^{7/8}\gamma\,{}_2F_1\left(\frac{1}{4},1;\frac{5}{4};\frac{i}{2}\right) \\ -\frac{16}{3}(-1)^{3/8}\,{}_2F_1\left(\frac{3}{4},1;\frac{7}{4};-2i\right)+\frac{16}{3}(-1)^{3/8}\gamma\,{}_2F_1\left(\frac{3}{4},1;\frac{7}{4};-2i\right)\end{aligned}$$

$$+ \frac{16}{3}(-1)^{7/8}\pi\, {}_2F_1\left(\frac{3}{4}, 1; \frac{7}{4}; -2i\right) + 8(-1)^{7/8}\log(4) \quad (38)$$

Proof. Use equation (35) set $n = -1, c = 1/2, b = -i, a = 1/4$ then take the first partial derivative with respect to k and set $k = 0$ and simplify.

Proposition 20. *Using equation (35) setting* $n = -2, c = 1, b = -i, a = 1/2, k = 1/2$ *and simplify to get*

$$\int_0^{+\infty} \frac{((x-6)x+1)\sqrt{\log\left(-\frac{i}{x}\right)}}{\sqrt{x}(x+1)^3} dx$$
$$= (1-i)\sqrt{\pi}\left(\sqrt{2}\Phi\left(-i, -\frac{1}{2}dx = \frac{1}{2}\right) + i\sqrt{2}\Phi\left(i, -\frac{1}{2}, \frac{3}{2}\right) + 1\right) \quad (39)$$

Proposition 21. *Using equation (35) setting* $n = -3, c = 1/2, b = -i, a = 1/3, k = -1/2$ *and simplify to get*

$$\int_0^{+\infty} \frac{x(x(4x-93)+120)-4}{x^{2/3}(x+2)^4\sqrt{\log\left(-\frac{i}{x}\right)}} dx$$
$$= \frac{1}{8}(-1)^{5/6}\sqrt{\pi}\left(27i\Phi\left(\frac{i}{2}, -\frac{5}{2}, \frac{4}{3}\right) + 2\left(\sqrt{3} + 54\Phi\left(-2i, -\frac{5}{2}, \frac{2}{3}\right)\right)\right) \quad (40)$$

Proposition 22. *Using equation (35) setting* $n = -1, c = 1, b = -i, a = 1/2, k = 3/2$ *and simplify to get*

$$\int_0^{+\infty} \frac{(x-1)\log^{\frac{3}{2}}\left(-\frac{i}{x}\right)}{\sqrt{x}(x+1)^2} dx$$
$$= \left(-\frac{3}{4} + \frac{3i}{4}\right)\sqrt{\pi}\left(\sqrt{2}\Phi\left(-i, \frac{3}{2}, \frac{1}{2}\right) + i\sqrt{2}\Phi\left(i, \frac{3}{2}, \frac{3}{2}\right) + 4\right) \quad (41)$$

10. Table of integrals

The examples displayed in this Table correspond to equations (12), (13), (15), (16), (20), (21), (22), (26), (27), (30), (32), and (34).

Table 1: Table of definite integrals

$f(x)$	$\int_0^{+\infty} f(x)dx$
$\frac{\tanh^{-1}(\alpha x)}{x(\beta+\log(x))^2}$	$\frac{1}{2}\left(\psi^{(0)}\left(-\frac{i(\beta-\log(\alpha))}{2\pi}\right)-\psi^{(0)}\left(\frac{-i\beta+i\log(\alpha)+\pi}{2\pi}\right)\right)$
$\frac{\left(\log^2(x)-\pi^2\right)\tanh^{-1}(x)}{x\left(\log^2(x)+\pi^2\right)^2}$	$\frac{1}{2}\left(\gamma+\psi^{(0)}\left(\frac{1}{2}\right)\right)$
$\frac{\tan^{-1}(dx)-\tan^{-1}(cx)}{x}$	$\frac{1}{2}\pi\log\left(\frac{d}{c}\right)$
$\frac{\log\left(\frac{b}{x}\right)\left(\tan^{-1}\left(\sqrt{d}\sqrt{x}\right)-\tan^{-1}\left(\sqrt{c}\sqrt{x}\right)\right)}{x}$	$-\frac{1}{4}\pi\log\left(\frac{c}{d}\right)\log\left(b^2cd\right)$
$\frac{1-x}{\sqrt{x}(x+1)^2\log(x)}$	$-\frac{4C}{\pi}$
$\frac{x-1}{\sqrt{x}(x+1)^2\sqrt{\log\left(-\frac{i}{x}\right)}}$	$\frac{(-1)^{3/4}\left(\zeta\left(\frac{3}{2},\frac{3}{8}\right)-\zeta\left(\frac{3}{2},\frac{7}{8}\right)\right)}{4\sqrt{\pi}}$
$\frac{(x-1)\sqrt{\log\left(-\frac{i}{x}\right)}}{\sqrt{x}(x+1)^2}$	$\sqrt[4]{-1}\sqrt{\pi}\left(\zeta\left(\frac{1}{2},\frac{7}{8}\right)-\zeta\left(\frac{1}{2},\frac{3}{8}\right)\right)$
$\frac{(cx-1)\log^k\left(\frac{b}{x}\right)}{(cx+1)^3}$	$\frac{ie^{-\frac{1}{2}i\pi k}(k-1)k(2\pi)^{k-1}\zeta\left(2-k,\frac{i\log(b)+i\log(c)+\pi}{2\pi}\right)}{c}$
$\frac{(x-1)\log(x)}{(x+1)^3\left(4\log^2(x)+\pi^2\right)}$	$\frac{\zeta\left(3,\frac{3}{4}\right)}{8\pi^2}$
$\frac{((x-4)x+1)\log\left(\log\left(\frac{1}{x}\right)\right)}{(x+1)^4}$	$\frac{7\zeta(3)}{2\pi^2}$
$\frac{(x-1)((x-10)x+1)\log\left(\frac{1}{x}\right)\log\left(\log\left(\frac{1}{x}\right)\right)}{(x+1)^5}$	$-\frac{7\zeta(3)}{2\pi^2}$
$\frac{(x-1)((x-22)x+1)\log\left(\frac{1}{x}\right)\log\left(\log\left(\frac{1}{x}\right)\right)}{8\sqrt{x}(x+1)^4}$	$-\frac{2C}{\pi}$
$\frac{((x-6)x+1)\log\left(-\log(x)+\frac{i\pi}{2}\right)}{\sqrt{x}(x+1)^3}$	$\frac{\psi^{(1)}\left(\frac{3}{8}\right)-\psi^{(1)}\left(\frac{7}{8}\right)}{2\pi}$

11. Discussion

In this work the authors used their contour integral method and derived a definite integral using the Lerch function in terms of the Lerch function which has not been given before. A definite integral representation involving the Lerch function was also derived for the Lerch transformation. A table of integrals was produced for easy reading by interested readers. We will be using our contour integral method to derive other integrals in our future work.

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