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Tensor Product and Certain Solutions of Fractional Wave Type Equation

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Abstract. In this paper we find certain solutions of some fractional partial differential equations. Tensor product of Banach spaces is used to find some solutions where separation of variables does not work. We solve the fractional wave type equation using fractional Fourier series

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1. Introduction

In [7], a definition of the so-called $\alpha-$conformable fractional derivative was introduced: Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to \mathbb{R}$. For $x \in E$, let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists, then it is called the $\alpha-$**conformable fractional derivative** of f at x. If f is $\alpha-$differentiable on $(0, r)$ for some $r > 0$, and $\lim_{x\to 0^+} D^{\alpha} f(x)$ exists then we define

$$D^{\alpha} f(0) = \lim_{x \to 0} D^{\alpha} f(x).$$

For $\alpha \in (0,1]$ and f, g are $\alpha-$differentiable at a point t, one can easily see that the conformable derivative satisfies:

1. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, for all $a, b \in \mathbb{R}$.
2. $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
3. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$.

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4. $D^\alpha(\frac{f}{g}) = \frac{gD^\alpha(f)-fD^\alpha(g)}{g^2}$, $g(t) \neq 0$.

We list here the fractional derivatives of certain functions,

(i) $D^\alpha(t^p) = p\, t^{p-\alpha}$.

(ii) $D^\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$.

(iii) $D^\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$.

(iv) $D^\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives. Further, one should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$, then $D^{\frac{1}{2}}(f)(0) = 1$. This is not the case for the known classical fractional derivatives, since $D^1(f)(0)$ does not exist. For more on fractional calculus and its applications we refer to [7]-[6].

Many differential equations can be transformed to fractional form and can have many applications in many branches of science. The main technique to solve partial differential equations is using Fourier series. So, fractional Fourier series was introduced in [3]. Such a concept proved to be very fruitful in solving fractional partial differential equations.

In this paper we will use fractional Fourier series to solve a fractional wave type equation. In Section 2 we introduce the atomic solution. The complete solution is given in Section 3.

2. Atomic solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by

$$T(x^*) = x^*(x)y$$

is a bounded one rank linear operator. We write $x \otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products. Atoms are used in theory of best approximation in Banach spaces, see [2]. One of the known results, see [4], that we need in our paper is that: If the sum of two atoms is an atom, then either the first components are dependent or the second ones are dependent. For more on tensor products of Banach spaces, we refer to [4].

Let us write $D_x^\alpha u$ to mean the partial $\alpha-$derivative of u with respect to x. Further we write $D_x^{2\alpha}u$ to mean $D_x^\alpha D_x^\alpha u$. Similarly for derivatives with respect to y.

If f is a function of one variable, say x , we write f^α, $f^{2\alpha}$ to denote $D_x^\alpha f$ and $D_x^{2\alpha} f$ respectively.

Our main object in this section is to find an atomic solution of the equation

$$D_x^{2\alpha} D_y^{2\beta} u + D_x^{\alpha} D_y^{\beta} u = 2u\,, \tag{1}$$

where by an atomic solution we mean a solution of the form $u(x,y) = P(x)Q(y)$.

Remark 1. One should remark that not every linear partial differential equation (fractional or not) can be solved using separation of variables. In such a case, the concept of atomic solution is inevitable. In equation (1), the method of separation of variables is not possible though the equation is linear. Hence we try to find an atomic solution of this equation. In other words, we look for a solution of the form

$$u(x,y) = P(x)Q(y).$$

Procedure

Let $u(x,y) = P(x)Q(y)$. Substitute in equation (1) to get:

$$P^{2\alpha}(x)Q^{2\beta}(y) + P^{\alpha}(x)Q^{\beta}(y) = 2P(x)Q(y).$$

This can written in tensor product form as:

$$P^{2\alpha} \otimes Q^{2\beta} + P^{\alpha} \otimes Q^{\beta} = P \otimes 2Q\,. \tag{2}$$

Let us consider the following conditions : $P(0) = 0$, $P^{\alpha}(0) = 1$.

In equation (2), we have the situation: the sum of two atoms is an atom.

Hence we have two cases:

Case(i): $P^{2\alpha} = P^{\alpha}$. Using the result in [5], we get

$$P(x) = e^{\frac{x^{\alpha}}{\alpha}}\,. \tag{3}$$

Now, we substitute in (2) to get

$$e^{x} \otimes [Q^{2\beta} + Q^{\beta}] = e^{x} \otimes Q\,.$$

Hence, $Q^{2\beta} + Q^{\beta} = 2Q$. Again, using the result in [5],

$$Q(y) = c_1 e^{-2\frac{y^{\beta}}{\beta}} + c_2 e^{\frac{y^{\beta}}{\beta}}$$

Using the conditions $Q(0) = Q^{\beta}(0) = 1$, we get

$$Q(y) = -\frac{1}{3} e^{-2\frac{y^{\beta}}{\beta}} + \frac{1}{3} e^{\frac{y^{\beta}}{\beta}}\,. \tag{4}$$

From (3) and (4), we obtain the atomic solution of (1) as:

$$u(x,y) = e^{\frac{x^{\alpha}}{\alpha}} \left(-\frac{1}{3} e^{-2\frac{y^{\beta}}{\beta}} + \frac{1}{3} e^{\frac{y^{\beta}}{\beta}}\right). \tag{5}$$

One can easily check that the atom u in (5) satisfies (1).

Case (ii): $Q^{2\beta} = Q^{\beta}$. Following the same steps as in case (i), we find the atomic solution in the form

$$u(x,y) = (-\frac{1}{3}e^{-2\frac{x^\alpha}{\alpha}} + \frac{1}{3}e^{\frac{x^\alpha}{\alpha}})e^{\frac{y^\beta}{\beta}}$$

3. Complete Solution

Consider the fractional partial differential equation

$$D_x^{2\beta}u - c^2 D_y^{2\alpha}u = D_y^{\alpha}u \tag{6}$$

with conditions

$$u(x,0) = f(x)\ ,\ u(x,1) = 0,\ u(L,y) = 0,\ u(0,y) = 0\,, \quad 0 < \alpha, \beta < 1.$$

Here c is a given constant. This is called fractional wave type equation. We will use fractional Fourier series and separation of variables to solve equation (6).

Remark 2. One may attempt to use change of variables to transform it to an ordinary partial differential equation. This is possible if in equation (1) and (6), the function u is $u = u(\frac{x^\alpha}{\alpha}, \frac{y^\alpha}{\alpha})$. But the function u in equations in (1) and (6) is $u = u(x,y)$. So any change of variables will not simplify the problem. Further, the partial derivatives of u: u_x and u_y need not to be exist even if $D_x^\alpha u$ and $D_y^\alpha u$ exist, see [7].

Let

$$u(x,y) = P(x)Q(y)\,.$$

Substitute in the equation (6) to get

$$P^{2\beta}(x)Q(y) - c^2P(x)Q^{2\alpha}(y) = P(x)Q^{\alpha}(y)\,.$$

Simplifying to get

$$P^{2\beta}(x)Q(y) - c^2P(x)Q^{2\alpha}(y) - P(x)Q^{\alpha}(y) = 0\,,$$

$$P^{2\beta}(x)Q(y) - P(x)\left(c^2Q^{2\alpha}(y) + Q^{\alpha}(y)\right) = 0\,,$$

$$P^{2\beta}(x)Q(y) = P(x)\left(c^2Q^{2\alpha}(y) + Q^{\alpha}(y)\right)\,.$$

From which we obtain

$$\frac{P^{2\beta}(x)}{P(x)} = \frac{c^2Q^{2\alpha}(y) + Q^{\alpha}(y)}{Q(y)} = \lambda\,.$$

Since x and y are independent variables, then we get

$$\frac{P^{2\beta}(x)}{P(x)} = \lambda$$

and

$$\frac{c^2 Q^{2\alpha}(y) + Q^{\alpha}(y)}{Q(y)} = \lambda \,.$$

Simplifying to get

$$P^{2\beta}(x) - \lambda P(x) = 0 \tag{7}$$

and

$$c^2 Q^{2\alpha}(y) + Q^{\alpha}(y) - \lambda Q(y) = 0 \,. \tag{8}$$

Let us first deal with equation (7). There are three possibilities for λ:

Case 1: $\lambda = 0$

Then equation (7) becomes $P^{2\beta}(x) = 0$. Using the result in [1], we see that $P(x) = c_1 \frac{x^\beta}{\beta} + c_2$. By using the condition $u(0, y) = 0$, we get $c_2 = 0$. Another use of condition $u(L, y) = 0$ we get $c_1 = 0$. So, $P(x) = 0$. Thus $\lambda = 0$ gives the trivial solution.

Case 2: $\lambda = \mu^2 > 0$

Then equation (7) becomes

$$P^{2\beta}(x) = \mu^2 P(x) \,.$$

Using the result in [5], we see that

$$P(x) = c_1 e^{\mu \frac{x^\beta}{\beta}} + c_2 e^{-\mu \frac{x^\beta}{\beta}} \,.$$

Using the condition $u(0, y) = 0$, we get $c_1 = -c_2$. So, $P(x) = 2c_1 \sinh(\mu \frac{x^\beta}{\beta})$. Another use of condition $u(L, y) = 0$ we get $2c_1 \sinh(\mu \frac{L^\beta}{\beta}) = 0$. Hence, $c_1 \neq 0$ and so $\mu = 0$. Thus, $P(x) = 0$. Therefore $\lambda > 0$ gives the trivial solution.

Case 3: $\lambda = -\mu^2 < 0$

Then equation (7) becomes

$$P^{2\beta}(x) + \mu^2 P(x) = 0 \,.$$

Using results in [5], we get

$$P(x) = c_1 \cos(\mu \frac{x^\beta}{\beta}) + c_2 \sin(\mu \frac{x^\beta}{\beta}).$$

Applying the condition $u(0, y) = 0$ we get $c_1 = 0$. So, $P(x) = c_2 \sin(\mu \frac{x^\beta}{\beta})$. Another use of condition $u(L, y) = 0$ gives $c_2 \sin(\mu \frac{L^\beta}{\beta}) = 0$. Then $c_2 \neq 0$ and so $\sin(\mu \frac{L^\beta}{\beta}) = 0$. Hence,

$$\mu = n\pi \frac{\beta}{L^\beta} \,. \tag{9}$$

So,

$$P(x) = c_n \sin(n\pi \frac{x^\beta}{L^\beta}) \,, \quad n = 1, 2, \ldots \,. \tag{10}$$

Now, we go back to equation (8). Substituting the value of μ that we got in (9), equation (8) becomes

$$c^2Q^{2\alpha}(y) + Q^{\alpha}(y) + \mu^2 Q(y) = 0\,.$$

Another use of the result in [5], we get two cases under consideration:

Case i: $1 - 4\mu^2c^2 > 0$.

$\mu^2 < 1$, $\mid \mu \mid < 1$, $1 - 4\mu^2c^2 = (\sqrt{1-4\mu^2c^2})^2$. Then we get,

$$r = \frac{-1 \pm \sqrt{1-4\mu^2c^2}}{2c^2}\,.$$

So

$$Q(y) = c_1 e^{\frac{-1+\sqrt{1-4\mu^2c^2}}{2c^2}\frac{y^\alpha}{\alpha}} + c_2 e^{\frac{-1-\sqrt{1-4\mu^2c^2}}{2c^2}\frac{y^\alpha}{\alpha}}\,.$$

Using condition $u(x,0) = 0$, we get $c_1 = -c_2$. So,

$$Q(y) = 2c_1 \sinh\left(\frac{-1+\sqrt{1-4\mu^2c^2}}{2c^2}\frac{y^\alpha}{\alpha}\right)\,. \tag{11}$$

Thus, combining (10) and (11) we get:

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(n\pi\frac{x^\beta}{L^\beta}\right)\sinh\left(\frac{-1+\sqrt{1-4(\frac{n\beta\pi}{L^\beta})^2c^2}}{2c^2}\frac{y^\alpha}{\alpha}\right)\,.$$

By using $D_y^\beta(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} b_n \left(\frac{-1+\sqrt{1-4(\frac{n\beta\pi}{L^\beta})^2c^2}}{2c^2}\right)\sin\left(n\pi\frac{x^\beta}{L^\beta}\right)\,.$$

So

$$b_n = \frac{2\beta}{P^\beta\left(\frac{-1+\sqrt{1-4(\frac{n\beta\pi}{L^\beta})^2c^2}}{2c^2}\right)}\int_0^P f(x)\sin\left(n\pi\frac{x^\beta}{L^\beta}\right)\frac{dx}{x^{1-\beta}}\,.$$

Case ii: $1 - 4\mu^2c^2 < 0$.

$\mu^2 > 1$, $\mid \mu \mid > 1$, $1 - 4\mu^2c^2 = -(4\mu^2c^2 - 1) = (\sqrt{4\mu^2c^2-1}^2)$. Then we get,

$$r = \frac{-1 \pm i\sqrt{4\mu^2c^2-1}}{2c^2}\,.$$

So

$$Q(y) = c_1\cos\left(\frac{-1+\sqrt{4\mu^2c^2-1}}{2c^2}\frac{y^\alpha}{\alpha}\right) + c_2\sin\left(\frac{-1+\sqrt{4\mu^2c^2-1}}{2c^2}\frac{y^\alpha}{\alpha}\right)$$

By using condition $u(x,0)=0$ we get $c_1=0$. Then,

$$Q(y)=c_2\sin\left(\frac{-1+\sqrt{4\mu^2c^2-1}}{2c^2}\frac{y^\alpha}{\alpha}\right). \tag{12}$$

Thus, (10) and (12) gives

$$u(x,y)=\sum_{n=1}^{\infty}b_n\sin\left(n\pi\frac{x^\beta}{L^\beta}\right)\sin\left(\frac{-1+\sqrt{4\mu^2c^2-1}}{2c^2}\frac{y^\alpha}{\alpha}\right). \tag{13}$$

By using condition $u_y(x,0)=f(x)$ we deduce that

$$f(x)=\sum_{n=1}^{\infty}b_n\left(\frac{-1+\sqrt{4(\frac{n\beta\pi}{L^\beta})^2c^2-1}}{2c^2}\right)\sin\left(n\pi\frac{x^\beta}{L^\beta}\right).$$

Hence, using [3], we find

$$b_n=\frac{2\beta}{P^\beta\left(\frac{-1+\sqrt{4(\frac{n\beta\pi}{L^\beta})^2c^2-1}}{2c^2}\right)}\int_0^P f(x)sin(n\pi\frac{x^\beta}{L^\beta})\frac{dx}{x^{1-\beta}}$$

where P is a period of the function f which equals to $(2\beta\pi)^{\frac{1}{\beta}}$. So we got the complete solution of the differential equations (6).

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