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Note on $(i,j)-m_X-\beta-$Exterior Sets in Biminimal Structure Spaces

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Abstract. In that paper, the concept of $(i,j)-m_X-\beta-$exterior sets in a biminimal structure space (BSS) and a biminimal structure subspace (BSs) were introduced. Based on properties of BSS and BSs, some new notions and several properties of those sets dealing with this space were obtained in both of BSS and BSs. Some examples were given to illustrate the effectiveness of these results.

2020 Mathematics Subject Classifications: 22A05, 22A15

Key Words and Phrases: Exterior sets, Biminimal structure spaces, $(i,j)-m_X-\beta-$exterior sets,$(i,j)-m_X-\beta-$closed sets,$(i,j)-m_X-\beta-$open sets

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1. Introduction

A general space in mathematics, topology space has been widely studied in every field of mathematics as a fundamental concept including the definition of limits, continuity, neighborhoods, closed sets, open set and connnectedness among others. In 2020, T. M. Al-shami et. al. [1] focused their attention on topology space. The concept of supra semi limit points of a set and new types of separation axioms using supra semi-open sets were introduced to minimize the conditions of topology for other reasons. Some applications of supra preopen sets on supra topological spaces was studied by M. E. El-Shafei and et. al [5]. The concept of supra prehomeomorphism maps and the concepts of supra limit and supra boundary points of a set with respect to supra preopen sets and their properties were introduced. More recently, A. Mhemdi and T. M. Al-shami [6] defined the functional separation axioms on general topology and provided some notions of them. The notions of almost SD-compact and almost SD-Lindelöf spaces, nearly SD-compact and nearly SD-Lindelöf spaces, and mildly SD-compact and mildly SD-Lindelöf spaces were investigated by T. M. Al-shami and T. Noiri [13].

The above discussion motivated the current study, of some branches of topology. The

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concept of a biminimal structure space (BSS) and some properties of $m^1_X m^2_X-$closed sets and $m^1_X m^2_X-$open sets in BSSs were introduced by Boonpok [2] in 2010. That is, (X, m^1_X, m^2_X) is called a biminimal structure space, where X is a nonempty set and m^1_X, m^2_X are minimal structures on X where minimal structures are defined by giving $P(X)$ as the power set of a nonempty set X. A subfamily m_X of $P(X)$ is called a minimal structure on X if $\emptyset \in m_X$ and $X \in m_X$.

Biminimal structure space has been of wide interest in studying in Topology. Furthermore, Boonpok et al. [2–4] provided some properties of them to as a preliminary for the current study, such as $(i,j)-m_X-\alpha-$closed, $(i,j)-m_X-\alpha-$open, $(i,j)-m_X-\beta-$closed and $(i,j)-m_X-\beta-$open, which are advantageous for studying BSS. Later, S. Sompong and S. Muangchan [10, 11] studied the notion of exterior sets in this space and obtained some characterizations and fundamental properties of those sets. E. Subha and N. Nagaveni [12] studied strongly minimal generalized closed set in BSSs and obtained some properties for the set. Later, P. Prasertsang and S. Sompong [8, 9] studied the concept of $(i,j)-m_X-\alpha-$boundary and exterior sets and $(i,j)-m_X-\beta-$boundary sets and provided some fundamental properties of such sets dealing with those spaces as well, which was relevant to the current research.

In this paper, the concepts of $(i,j)-m_X-\beta-$exterior sets are introduced and some fundamental properties of those sets are obtained and some examples are given for completing some properties. Lastly, the special properties of a biminimal structure subspace and the product of those sets are defined and then some fundamental properties are provided.

2. Preliminaries

In this section we recall some notions, notations and previous results.

Definition 1. *[11] Let $P(X)$ be the power of nonempty set X. A subfamily m_X of $P(X)$ is called a minimal structure (briefly $m-$ structure) on X if $\emptyset \in m_X$ and $X \in m_X$.*

Definition 2. *[2] Let X be a nonempty set and m^1_X, m^2_X be minimal structures on X. The triple (X, m^1_X, m^2_X) is called a biminimal structure space (briefly BSS) or a bispace (briefly bi $m-$space [7])*

Lemma 1. *[3] Let (X, m^1_X, m^2_X) be a biminimal structure space and A be a subset of X. It follows that:*

1. *A is $(i,j)-m_X-$regular $-$closed if and only if $A = m^i_X Cl(m^j_X Int(A))$,*
2. *A is $(i,j)-m_X-$semi$-$closed if and only if $m^i_X Int(m^j_X Cl(A)) \subseteq A$,*
3. *A is $(i,j)-m_X-$preclosed if and only if $m^i_X Cl(m^j_X Int(A)) \subseteq A$,*
4. *A is $(i,j)-m_X-\alpha-$closed if and only if $m^i_X Cl(m^j_X Int(m^i_X Cl(A))) \subseteq A$,*
5. *A is $(i,j)-m_X-\beta-$closed if and only if $m^i_X Int(m^j_X Cl(m^i_X Int(A))) \subseteq A$.*

Definition 3. *[9] Let (X, m^1_X, m^2_X) be a biminimal structure space and A be a subset of X. Then, $m^{ij}_X-\beta-$closure of A and the $m^{ij}_X-\beta-$interior of A where $i,j = 1,2$ and $i \neq j$. are defined as follows:*

1. $m_X^{ij}Cl_{\mathscr{B}}(A) = \bigcap\{F : A \subseteq F, F \text{ is } (i,j) - m_X - \beta - closed\}$,
2. $m_X^{ij}Int_{\mathscr{B}}(A) = \bigcup\{U : U \subseteq A, U \text{ is } (i,j) - m_X - \beta - \text{ open}\}$.

Lemma 2. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure space and A, B be subsets of X, the following hold:*
1. *$m_X^{ij}Cl_{\mathscr{B}}(\emptyset) = \emptyset$, $m_X^{ij}Cl_{\mathscr{B}}(X) = X$, $m_X^{ij}Int_{\mathscr{B}}(\emptyset) = \emptyset$ and $m_X^{ij}Int_{\mathscr{B}}(X) = X$,*
2. *$A \subseteq m_X^{ij}Cl_{\mathscr{B}}(A)$ and $m_X^{ij}Int_{\mathscr{B}}(A) \subseteq A$,*
3. *If $A \subseteq B$ then $m_X^{ij}Cl_{\mathscr{B}}(A) \subseteq m_X^{ij}Cl_{\mathscr{B}}(B)$ and $m_X^{ij}Int_{\mathscr{B}}(A) \subseteq m_X^{ij}Int_{\mathscr{B}}(B)$.*

Lemma 3. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. The following properties hold:*
1. *$m_X^{ij}Cl_{\mathscr{B}}(A)$ is $(i,j) - m_X - \beta-$closed,*
2. *$m_X^{ij}Int_{\mathscr{B}}(A)$ is $(i,j) - m_X - \beta-$open,*
3. *A is $(i,j) - m_X - \beta-$closed if and only if $m_X^{ij}Cl_{\mathscr{B}}(A) = A$,*
4. *A is $(i,j) - m_X - \beta-$open if and only if $m_X^{ij}Int_{\mathscr{B}}(A) = A$.*

Lemma 4. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure space and A, B be subsets of X, the following hold:*
1. *If A and B are $(i,j) - m_X - \beta-$closed then $A \cap B$ is $(i,j) - m_X - \beta-$closed,*
2. *If A and B are $(i,j) - m_X - \beta-$open then $A \cup B$ is $(i,j) - m_X - \beta-$open.*

Lemma 5. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure space and A a subset of X:*
1. *$m_X^{ij}Int_{\mathscr{B}}(X \backslash A) = X \backslash m_X^{ij}Cl_{\mathscr{B}}(A)$,*
2. *$m_X^{ij}Cl_{\mathscr{B}}(X \backslash A) = X \backslash m_X^{ij}Int_{\mathscr{B}}(A)$.*

Lemma 6. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure space and A a subset of X, for any $i, j = 1, 2$ and $i \neq j$,*
1. *$m_X^{ij}Bdr_{\mathscr{B}}(A) \cap m_X^{ij}Int_{\mathscr{B}}(X \backslash A) = \emptyset$,*
2. *$m_X^{ij}Cl_{\mathscr{B}}(X \backslash A) = m_X^{ij}Bdr_{\mathscr{B}}(A) \cup m_X^{ij}Int_{\mathscr{B}}(A)$,*
3. *$X = m_X^{ij}Int_{\mathscr{B}}(A) \cup m_X^{ij}Bdr_{\mathscr{B}}(A) \cup m_X^{ij}Int_{\mathscr{B}}(X \backslash A)$ is a pairwise disjoint union.*

Definition 4. *[9] Let (X, m_X^1, m_X^2) be a biminimal structure spaces and W be a subset of X. Define m_W^1 and m_W^2 as follows: $m_W^1 = A \cap W : A \in m_X^1$ and $m_Y^2 = B \cap W : B \in m_X^2$. A triple (W, m_W^1, m_W^2) is called a biminimal structure subspace of (X, m_X^1, m_X^2).*

Let (W, m_W^1, m_W^2) be a biminimal structure subspace of (X, m_X^1, m_X^2), and A be a subset of W. The $(i,j) - m_W - \beta-$closure and $(i,j) - m_W - \beta-$interior of A with respect to m_W^{ij} are denoted by $m_W^{ij}Cl_{\mathscr{B}}(A)$ and $m_W^{ij}Int_{\mathscr{B}}(A)$, respectively (for $i = 1, 2$ and $i \neq j$). Then, $m_W^{ij}Cl_{\mathscr{B}}(A) = W \cap m_X^{ij}Cl_{\mathscr{B}}(A)$,

$m_W^{ij}Int_{\mathscr{B}}(A) = W \cap m_X^{ij}Int_{\mathscr{B}}(A)$

and $m_W^{ij}Bdr_{\mathscr{B}}(A) = W \cap m_X^{ij}Bdr_{\mathscr{B}}(A)$

consequently, $m_W^{ij}Ext_{\mathscr{B}}(A) = W \cap m_X^{ij}Ext_{\mathscr{B}}(A)$.

3. Main Results

In this section, we introduce the concepts of $(i,j)-m_X-\beta-$exterior sets in biminimal structure space which contains some characterizations and several fundamental properties of those sets.

Definition 5. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X and $x \in X$. Then, x is called $(i,j)-m_X-\beta-$exterior point of A if $x \in m_X^{ij}Int_{\mathscr{B}}(X\backslash A)$. The set of all $(i,j)-m_X-\beta-$exterior point of A are denoted by: $m_X^{ij}Ext_{\mathscr{B}}(A)$ where $i,j=1,2$ and $i \neq j$.*

By the Definition 5, $m_X^{ij}Ext_{\mathscr{B}}(A) = m_X^{ij}Int_{\mathscr{B}}(X\backslash A) = X\backslash m_X^{ij}Cl_{\mathscr{B}}(A)$.

Example 1. *Let $X = \{1,2,3\}$. Define $m-$structures m_X^1 and m_X^2 on the biminimal structure space X as follows:*
$m_X^1 = \{\emptyset, \{2\}, \{1,3\}, X\}$ *and* $m_X^2 = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}, X\}$.
We have that: $m_X^{12}Ext_{\mathscr{B}}(\{1,2\}) = \{3\}$ *and* $m_X^{21}Ext_{\mathscr{B}}(\{1,2\}) = \emptyset$.

Lemma 7. *Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X. Then, for any $i,j=1,2$ and $i \neq j$, the following statements hold:*
1. $m_X^{ij}Ext_{\mathscr{B}}(\emptyset) = X$ *and* $m_X^{ij}Ext_{\mathscr{B}}(X) = \emptyset$,
2. $m_X^{ij}Ext_{\mathscr{B}}(A) \cap A = \emptyset$ *and* $m_X^{ij}Ext_{\mathscr{B}}(A) \cap m_X^{ij}Cl_{\mathscr{B}}(A) = \emptyset$,
3. $m_X^{ij}Ext_{\mathscr{B}}(A) \cap m_X^{ij}Ext_{\mathscr{B}}(X\backslash A) = \emptyset$ *and* $m_X^{ij}Ext_{\mathscr{B}}(A) \cap m_X^{ij}Bdr_{\mathscr{B}}(A) = \emptyset$,
4. $X = m_X^{ij}Int_{\mathscr{B}}(A) \cup m_X^{ij}Bdr_{\mathscr{B}}(A) \cup m_X^{ij}Ext_{\mathscr{B}}(A)$ *is a pairwise disjoint union.*

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X.

1. Since $m_X^{ij}Cl_{\mathscr{B}}(\emptyset) = \emptyset$ and $m_X^{ij}Cl_{\mathscr{B}}(X) = X$, we obtain: $m_X^{ij}Ext_{\mathscr{B}}(\emptyset) = X\backslash\emptyset = X$ and $m_X^{ij}Ext_{\mathscr{B}}(X) = X\backslash X = \emptyset$.

2. By Lemma 2 (2), $X\backslash m_X^{ij}Cl_{\mathscr{B}}(A) \subseteq X\backslash A, (X\backslash m_X^{ij}Cl_{\mathscr{B}}(A)) \cap A \subseteq \emptyset$. That is: $m_X^{ij}Ext_{\mathscr{B}}(A) \cap A = \emptyset$. It follow that: $m_X^{ij}Ext_{\mathscr{B}}(A) \cap m_X^{ij}Cl_{\mathscr{B}}(A) = \emptyset$.

3. It follows by Lemma 6 (1).

4. It is obvious by Definition 5 and Lemma 6 (2), (3).

Theorem 1. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A, B be subsets of X with $A \subseteq B$. Then, for $i,j=1,2$ and $i \neq j$,*
1. $m_X^{ij}Ext_{\mathscr{B}}(B) \subseteq m_X^{ij}Ext_{\mathscr{B}}(A)$,
2. $m_X^{ij}Ext_{\mathscr{B}}(B) \subseteq X\backslash m_X^{ij}Bdr_{\mathscr{B}}(A)$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A, B are subsets of X with $A \subseteq B$. For any $i,j=1,2$ and $i \neq j$,

1. From Lemma 2 (3), $m_X^{ij}Cl_{\mathscr{B}}(A) \subseteq m_X^{ij}Cl_{\mathscr{B}}(B)$ yields

$$X\backslash m_X^{ij}Cl_{\mathscr{B}}(B) \subseteq X\backslash m_X^{ij}Cl_{\mathscr{B}}(A),$$
$$m_X^{ij}Ext_{\mathscr{B}}(B) \subseteq m_X^{ij}Ext_{\mathscr{B}}(A).$$

2. By Lemma 7 (3), $m_X^{ij}Ext_{\mathscr{B}}(A) \subseteq X\backslash m_X^{ij}Bdr_{\mathscr{B}}(A)$ and by (1), we have
$m_X^{ij}Ext_{\mathscr{B}}(B) \subseteq X\backslash m_X^{ij}Bdr_{\mathscr{B}}(A)$.

Corollary 1. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be subsets of X. Then, for $i, j = 1, 2$ and $i \neq j$,*
1. $m_X^{ij}Ext_{\mathscr{B}}(A) \subseteq m_X^{ij}Ext_{\mathscr{B}}(m_X^{ij}Int_{\mathscr{B}}(A))$,
2. $m_X^{ij}Ext_{\mathscr{B}}(m_X^{ij}Cl_{\mathscr{B}}(A)) \subseteq m_X^{ij}Ext_{\mathscr{B}}(A)$,
3. $m_X^{ij}Ext_{\mathscr{B}}(A) \subseteq X\backslash m_X^{ij}Bdr_{\mathscr{B}}(m_X^{ij}Int_{\mathscr{B}}(A))$,
4. $m_X^{ij}Ext_{\mathscr{B}}(m_X^{ij}Cl_{\mathscr{B}}(A)) \subseteq X\backslash m_X^{ij}Bdr_{\mathscr{B}}(A)$.

Proof. It follows by Theorem 1 and Lemma 2 (2).

Theorem 2. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. Then, for any $i, j = 1, 2$ and $i \neq j$, the following statement are true:*
1. A is $(i,j) - m_X - \beta-$closed if and only if $m_X^{ij}Ext_{\mathscr{B}}(A) = X\backslash$,
2. A is $(i,j) - m_X - \beta-$open if and only if $m_X^{ij}Ext_{\mathscr{B}}(X\backslash A) = A$.

Proof. Assume that (X, m_X^1, m_X^2) is a biminimal structure space and A is a subset of X.

1. ($\Longrightarrow$) Suppose that A is $(i,j) - m_X - \beta-$*closed*. Then, $m_X^{ij}Ext_{\mathscr{B}}(A) = X\backslash m_X^{ij}Cl_{\mathscr{B}}(A) = X\backslash A$.

($\Longleftarrow$) Suppose that $m_X^{ij}Ext_{\mathscr{B}}(A) = X\backslash A$. It means that $X\backslash m_X^{ij}Cl_{\mathscr{B}}(A) = X\backslash A$. Since $A \subseteq m_X^{ij}Cl_{\mathscr{B}}(A)$, then $m_X^{ij}Cl_{\mathscr{B}}(A) = A$. Finally, A is $(i,j) - m_X - \beta-$closed.

2. ($\Longrightarrow$) Suppose that A is $(i,j) - m_X - \beta-$open. Then, $X\backslash A$ is $(i,j) - m_X - \beta-$closed. Using (1), $m_X^{ij}Ext_{\mathscr{B}}(X\backslash A) = X\backslash(X\backslash A) = A$.

($\Longleftarrow$) Suppose that $m_X^{ij}Ext_{\mathscr{B}}(X\backslash A) = A$. We have $A = X\backslash m_X^{ij}Cl_{\mathscr{B}}(X\backslash A)$
$= X\backslash(X\backslash m_X^{ij}Int_{\mathscr{B}}(A)) = m_X^{ij}Int_{\mathscr{B}}(A))$. Hence, A is $(i,j) - m_X - \beta-$open.

Corollary 2. *Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X. Then, for $i, j = 1, 2$ and $i \neq j$,*
1. $m_X^{ij}Ext_{\mathscr{B}}(m_X^{ij}Cl_{\mathscr{B}}(A)) = m_X^{ij}Ext_{\mathscr{B}}(A)$,
2. $m_X^{ij}Ext_{\mathscr{B}}(X\backslash m_X^{ij}Ext_{\mathscr{B}}(A)) = m_X^{ij}Ext_{\mathscr{B}}(A)$

Proof. This follows by Theorem 2 immediately.

From Example 1, $m_X^{12}Ext_{\mathscr{B}}(\{2\}) \cup m_X^{12}Ext_{\mathscr{B}}(\{1,3\}) \neq m_X^{12}Ext_{\mathscr{B}}(\{2\} \cap \{1,3\})$, whereas $m_X^{12}Ext_{\mathscr{B}}(\{2\}) \cup m_X^{12}Ext_{\mathscr{B}}(\{1,2\}) = m_X^{12}Ext_{\mathscr{B}}(\{2\} \cap \{1,\})$. Therefore, it needs some conditions to show that

$$m_X^{ij}Ext_{\mathscr{B}}(A) \cup m_X^{ij}Ext_{\mathscr{B}}(B) = m_X^{ij}Ext_{\mathscr{B}}(A \cap B),$$

which found in the next result. Similarly, the following equation

$$m_X^{ij}Ext_{\mathscr{B}}(A \cup B) = m_X^{ij}Ext_{\mathscr{B}}(A) \cap m_X^{ij}Ext_{\mathscr{B}}(B)$$

is true if it has some appropriate conditions.

Theorem 3. *Let (X, m^1_X, m^2_X) be a biminimal structure space, A, B be subsets of X. Then, for any $i, j = 1, 2$ and $i \neq j$, we have:*

1. *If A and B are $(i, j) - m_X - \beta-$closed, then*
$$m^{ij}_X Ext_{\mathscr{B}}(A) \cup m^{ij}_X Ext_{\mathscr{B}}(B) = m^{ij}_X Ext_{\mathscr{B}}(A \cap B).$$
2. *If A, B and $A \cup B$ are $(i, j) - m_X - \beta-$closed, then*
$$m^{ij}_X Ext_{\mathscr{B}}(A \cup B) = m^{ij}_X Ext_{\mathscr{B}}(A) \cap m^{ij}_X Ext_{\mathscr{B}}(B).$$

Proof. Assume that (X, m^1_X, m^2_X) is a biminimal structure space and A, B are subsets of X.

1. Assume that A and B are $(i, j) - m_X - \beta-$closed. Therefore, $A \cap B$ is also $(i, j) - m_X - \beta-$closed. By Theorem 2 (1),
$$\begin{aligned} m^{ij}_X Ext_{\mathscr{B}}(A \cap B) &= X \backslash A \cap B \\ &= (X \backslash A) \cup (X \backslash B) \\ &= m^{ij}_X Ext_{\mathscr{B}}(A) \cup m^{ij}_X Ext_{\mathscr{B}}(B). \end{aligned}$$

2. Assume that A, B and $A \cup B$ are $(i, j) - m_X - \beta-$closed. By Theorem 2 (1),$m^{ij}_X Ext_{\mathscr{B}}(A) = X \backslash A, m^{ij}_X Ext_{\mathscr{B}}(B) = X \backslash B$ and $m^{ij}_X Ext_{\mathscr{B}}(A \cup B) = X \backslash (A \cup B)$. Furthermore,
$$\begin{aligned} m^{ij}_X Ext_{\mathscr{B}}(A \cup B) &= X \backslash (A \cup B) \\ &= (X \backslash A) \cap (X \backslash B) \\ &= m^{ij}_X Ext_{\mathscr{B}}(A) \cap m^{ij}_X Ext_{\mathscr{B}}(B). \end{aligned}$$

Next, we give some notions for the product of two biminimal structure space as the following:

Definition 6. *Let (X, m^1_X, m^2_X) and (Y, m^1_Y, m^2_Y) be biminimal structure spaces, A, B be subsets of X and Y, respectively. Then,*
$$m^{ij}_{X \times Y} Cl_{\mathscr{B}}(A \times B) = [m^{ij}_X Cl_{\mathscr{B}}(A) \times Y] \cap [X \times m^{ij}_Y Cl_{\mathscr{B}}(B)]$$
where $i, j = 1, 2$ and $i \neq j$.

From Example 1, let $Y = \{a, b, c\}$, we have $m^1_Y = \{\emptyset, \{a\}, \{b\}, \{a, c\}, Y\}$ and $m^2_Y = \{\emptyset, \{c\}, \{a, b\}\{b, c\}, Y\}$. Therefore,
$m^{12}_{X \times Y} Cl_{\mathscr{B}}(\{2\} \times \{c\}) = [m^{12}_X Cl_{\mathscr{B}}(\{2\}) \times Y] \cap [X \times m^{12}_Y Cl_{\mathscr{B}}(\{c\})] = \{2, c\}.$

Lemma 8. *Let (X, m^1_X, m^2_X) and (Y, m^1_Y, m^2_Y) be biminimal structure spaces, A, B be subsets of X and Y, respectively. Then,*
$$m^{ij}_{X \times Y} Ext_{\mathscr{B}}(A \times B) = [m^{ij}_X Ext_{\mathscr{B}}(A) \times Y] \cup [X \times m^{ij}_Y Ext_{\mathscr{B}}(B)].$$

Proof. Let (Y, m^1_Y, m^2_Y) be a biminimal structure subspace of (X, m^1_X, m^2_X) and A a subset of Y. Let us consider:
$$\begin{aligned} m^{ij}_{X \times Y} Ext_{\mathscr{B}}(A \times B) &= (X \times Y) \backslash m^{ij}_{X \times Y} Cl_{\mathscr{B}}(A \times B) \\ &= (X \times Y) \backslash ((m^{ij}_X Cl_{\mathscr{B}}(A) \times Y) \cap (X \times m^{ij}_Y Cl_{\mathscr{B}}(B))) \\ &= ((X \times Y) \backslash (m^{ij}_X Cl_{\mathscr{B}}(A) \times Y)) \cup ((X \times Y) \backslash (X \times m^{ij}_Y Cl_{\mathscr{B}}(B))) \\ &= ((X \backslash m^{ij}_X Cl_{\mathscr{B}}(A)) \times Y) \cup (X \times (Y \backslash m^{ij}_Y Cl_{\mathscr{B}}(B)))) \\ &= (m^{ij}_X Ext_{\mathscr{B}}(A) \times Y) \cup (X \times m^{ij}_Y Ext_{\mathscr{B}}(B)). \end{aligned}$$

The next results study the biminimal structure subspace and obtain some properties of them. Furthermore, the product of the $(i, j) - m_X - \beta-$exterior sets is introduced.

Lemma 9. *Let* (W, m_W^1, m_W^2) *be a biminimal structure subspace of* (X, m_X^1, m_X^2)*,* A *and* B *are subsets of* X *and* W*, respectively, and* $A = B \cap W$ *are* $(i,j) - m_X - \beta -$ *closed. Then,* $m_W^{ij}Ext_{\mathscr{B}}(A) = m_X^{ij}Ext_{\mathscr{B}}(B) \cap W$*.*

Proof. Let (W, m_W^1, m_W^2) be a biminimal structure subspace of (X, m_X^1, m_X^2) and A be a subset of Y. Consider,

$$\begin{aligned} m_Y^{ij}Ext_{\mathscr{B}}(A) &= m_W^{ij}Ext_{\mathscr{B}}(B \cap W) \\ &= m_X^{ij}Ext_{\mathscr{B}}(B \cap W) \cap Y \\ &= [m_X^{ij}Ext_{\mathscr{B}}(B) \cup m_X^{ij}Ext_{\mathscr{B}}(W)] \cap Y \\ &= [m_X^{ij}Ext_{\mathscr{B}}(B) \cap W] \cup [m_X^{ij}Ext_{\mathscr{B}}(W) \cap W] \\ &= [m_X^{ij}Ext_{\mathscr{B}}(B) \cap W]. \end{aligned}$$

4. Conclusion

This study investigated $(i,j) - m_X - \beta-$exterior sets in a biminimal structure space (BSS) and a biminimal structure subspace (BSs). First, $(i,j) - m_X - \beta-$exterior sets in a biminimal structure space are defined in Definition 5. Second, Theorem 1 present the notion for the subsets of $(i,j) - m_X - \beta-$exterior sets. Third, the relation of $(i,j) - m_X - \beta-$exterior sets and $(i,j) - m_X - \beta-$closed and open sets are covered in Theorem 2. The union, intersection and product of $(i,j) - m_X - \beta-$exterior sets are given in Theorem 3 and Lemma 8. The authors describe the $(i,j) - m_X - \beta-$exterior sets of a biminimal structure subspace (BSs), in the last Section 3.

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