On 2-Resolving Sets in the Join and Corona of Graphs

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Abstract. Let $G$ be a connected graph. An ordered set of vertices $\{v_1, ..., v_l\}$ is a 2-resolving set in $G$ if, for any distinct vertices $u, w \in V(G)$, the lists of distances $(d_G(u, v_1), ..., d_G(u, v_l))$ and $(d_G(w, v_1), ..., d_G(w, v_l))$ differ in at least 2 positions. If $G$ has a 2-resolving set, we denote the least size of a 2-resolving set by $\dim_2(G)$, the 2-metric dimension of $G$. A 2-resolving set of size $\dim_2(G)$ is called a 2-metric basis for $G$. This study deals with the concept of 2-resolving set of a graph. It characterizes the 2-resolving set in the join and corona of graphs and determine the exact values of the 2-metric dimension of these graphs.

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1. Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater [10], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [4] where metric generators were called resolving sets. In [6], Monsanto, Acal and Rara discussed the strong resolving dominating sets in the join and corona of graphs while in [5], Monsanto and Rara discussed the resolving restrained domination in graphs.

Bailey and Yero in [1] demonstrated a construction of error-correcting codes from graphs by means of $k$-resolving sets, and present a decoding algorithm which makes use of covering designs.

The distance between two vertices $u$ and $v$ of a graph is the length of a shortest path
between $u$ and $v$, and we denote this by $d_G(u, v)$. In recent years, much attention has been paid to the metric dimension of graphs: this is the smallest size of a subset of vertices (called a resolving set) with the property that the list of distances from any vertex to those in the set uniquely identifies that vertex and is denoted by $\dim(G)$.

According to the paper of Saenpholphat et al. [9], for an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex $v$ in $G$, the $k$-vector (ordered $k$-tuple)

$$r(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))$$

is referred to as the (metric) representation of $v$ with respect to $W$. The set $W$ is called a resolving set for $G$ if distinct vertices have distinct representation with respect to $W$. Hence, if $W$ is a resolving set of cardinality $k$ for a graph $G$ of order $n$, then the set $\{r(v/W) : v \in V(G)\}$ consists of $n$ distinct $k$-vectors. A resolving set of minimum cardinality is called a minimum resolving set or a basis, and the cardinality of a basis for $G$ is the dimension $\dim(G)$ of $G$.

In the paper of Bailey et al. [1], an ordered set of vertices $W = \{w_1, ..., w_l\}$ is a $k$-resolving set for $G$ if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r(u/W)$ and $r(v/W)$ of $u$ and $v$, respectively, differ in at least $k$ positions. If $k = 1$, then the $k$-resolving set is called a resolving set for $G$. If $G$ has a $k$-resolving set, the minimum cardinality $\dim_k(G)$ is called the $k$-metric dimension of $G$.

In this paper, the concept of 2-resolving set in the join and corona of graphs is discussed.

### 2. Preliminary Results

In this study, we consider finite, simple and connected undirected graphs. For basic graph-theoretic concepts, we refer readers to [3].

**Remark 1.** Let $G$ be any connected graph of order $n \geq 2$. Then the vertex set of $G$ is a 2-resolving set in $G$. Hence, $2 \leq \dim_2(G) \leq n$.

**Proposition 1.** [7] \(\dim_2(G) = 2\) if and only if $G \cong P_n, n \geq 2$.

**Proposition 2.** For any complete graph $K_n$ of order $n \geq 2$, $\dim_2(K_n) = n$.

**Theorem 1.** Every 2-resolving set in a connected graph $G$ is a resolving set in $G$. Hence, $\dim(G) \leq \dim_2(G)$.

**Remark 2.** A superset of a 2-resolving set is a 2-resolving set.

**Remark 3.** Let $S \subseteq V(G)$. For any pair of vertices $x, y \in S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions. Hence, to prove that $S$ is a 2-resolving set in $G$, we only need to show that for every pair of vertices $x, y \in V(G)$ where $x \in S$ and $y \in V(G) \setminus S$ or both $x, y \in V(G) \setminus S$, $r(x/S)$ and $r(y/S)$ differ in at least 2 positions.
3. 2-Resolving Sets in the Join of Graphs

**Definition 1.** [2] The join $G + H$ of two graphs $G$ and $H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set

$$E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}.$$**

Note that the star $K_{1,n}$ can be expressed as the join of the trivial graph $K_1$ and the empty graph $\overline{K}_n$ of order $n$, that is, $K_{1,n} = K_1 + \overline{K}_n$. The graphs $F_n = K_1 + P_n$ and $W_n = K_1 + C_n$ of orders $n + 1$ are called fan and wheel, respectively.

**Definition 2.** Let $G = ((V(G), E(G))$ be a connected graph. The open neighborhood $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. Any element $u$ of $N_G(v)$ is called a neighbor of $v$.

The notation $x \in V(G)\setminus S$ means that $x \in V(G)$ but not in $S$.

**Definition 3.** Let $G$ be any nontrivial connected graph and $S \subseteq V(G)$. Then $S$ is a 2-locating set of $G$ if $\forall x, y \in V(G)$, $x \neq y$, the following are satisfied:

(i) If $x, y \in V(G) \setminus S$, then $\exists w, z \in S$, $w \neq z$ such that either:

(a) $w, z \in (N_G(x)) \setminus N_G(y)$, or

(b) $w, z \in (N_G(y)) \setminus N_G(x)$, or

(c) $w \in (N_G(x)) \setminus N_G(y)$ and $z \in (N_G(y)) \setminus N_G(x)$.

(ii) If $x \in S$, $y \in V(G) \setminus S$, then $\exists p \in (N_G(x) \cap S) \setminus N_G(y)$ or $p \in (N_G(y) \cap S) \setminus N_G(x)$.

The 2-locating number of $G$, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of $G$. A 2-locating set of $G$ of cardinality $ln_2(G)$ is referred to as $ln_2$-set of $G$.

**Example 1.** The sets $S_1 = \{c, d, e, f\}$ and $S_2 = \{a, b, c, f\}$ are 2-locating sets in $G$ in Figure 3. Moreover, $S_1$ and $S_2$ are $ln_2$-set in $G$. Thus, $ln_2(G) = |S_1| = |S_2| = 4$.

**Example 2.** Let $P_6 = [v_1, v_2, ..., v_6]$ be a path of order 6 and $S_1 = \{v_1, v_3, v_5, v_6\}$. Then $S_1$ is both 2-locating and 2-resolving set in $P_6$. On the other hand, $S_2 = \{v_2, v_4, v_6\}$ is a 2-resolving set but not 2-locating.

\[ln_2(G) = 4\]

Figure 1: A graph $G$ with $ln_2 = 4$
Example 3. For all \( n \geq 2 \), \( \ln_2(P_n) = \left\lfloor \frac{n+1}{2} \right\rfloor \).

Example 4. For all \( n \geq 5 \), \( \ln_2(C_n) = \left\lfloor \frac{n}{2} \right\rfloor \) and \( \ln_2(C_3) = 3 \), \( \ln_2(C_4) = 4 \).

Definition 4. Let \( G \) be any nontrivial connected graph and \( S \subseteq V(G) \). \( S \) is a strictly 2-locating (strictly 1-locating) set in \( G \) if \( S \) is 2-locating and \( |N_G(y) \cap S| \leq |S| - 2 \) for all \( y \in V(G) \). The strictly 2-locating (strictly 1-locating) number of \( G \), denoted by \( sln_2(G) \) (\( sln_1(G) \)), is the smallest cardinality of a strictly 2-locating (strictly 1-locating) set in \( G \). A strictly 2-locating (strictly 1-locating) set in \( G \) of cardinality \( sln_2(G) \) (\( sln_1(G) \)) is referred to as \( sln_2 \)-set (\( sln_1 \)-set) in \( G \).

Example 5. The set \( S_2 = \{a, b, c, f\} \) is a strictly 1-locating set in \( G \) in Figure 3. Moreover, \( S_2 \) is a \( sln_1 \)-set in \( G \). Thus, \( sln_1(G) = 4 \).

Example 6. The set \( S = \{u_1, u_3, u_5, u_7\} \) is a strictly 2-locating set in \( P_7 \) in Figure 3. Moreover, \( S \) is a \( sln_2 \)-set in \( P_7 \). Thus, \( sln_2(P_7) = 4 \).

\[ G: (-3,0)0\{c\}(u_1) (-3,-2) [c] (u_2) (-2,-2) [c] (u_3) (-1,-2) [c] (u_4) (0,-2) [c] (u_5) (1,-2) [c] (u_6) (2,-2) [c] (u_7) (3,-2) \]

Figure 2: A graph \( P_7 \) with \( sln_2 = 4 \)

Example 7. For all \( n \geq 4 \), \( sln_1(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{n is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{n is odd} \end{cases} \)

Example 8. For all \( n \geq 5 \), \( sln_1(C_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{n is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{n is odd} \end{cases} \)

Example 9. For all \( n \geq 6 \), \( sln_2(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{n is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{n is odd} \end{cases} \)

Example 10. For all \( n \geq 7 \), \( sln_2(C_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor, & \text{n is even} \\ \left\lceil \frac{n}{2} \right\rceil, & \text{n is odd} \end{cases} \)

Remark 5. Every strictly 2-locating set in \( G \) is strictly 1-locating. However, strictly 1-locating set in \( G \) need not be a strictly 2-locating set in \( G \).

Theorem 2. A proper subset \( S \) of \( V(K_1 + \overline{K}_n) \) is a 2-resolving set in \( K_1 + \overline{K}_n \) if and only if \( S = V(\overline{K}_n), \forall n \geq 2 \).
Proof. Let $S$ be a proper subset of $V(K_1 + \overline{K}_n)$. Suppose $S$ is a 2-resolving set in $K_1 + \overline{K}_n$ and suppose $\exists x \in V(\overline{K}_n) \setminus S$. Then $r(x/S)$ and $r(y/S)$ differ in at most one position for each $y \in V(\overline{K}_n)$. Thus, $S = V(\overline{K}_n)$.

Conversely, let $S = V(\overline{K}_n)$ and $x \in V(K_1)$. Then, $r(x/S) = (1, \ldots, 1)$ and $r(y/S) = (\ldots, 2, 2, 0, 2, \ldots)$ for each $y \in V(\overline{K}_n)$. Thus, $r(x/S)$ and $r(y/S)$ differ in at least two positions. Therefore $S$ is a 2-resolving set of $K_1 + \overline{K}_n$.

\[ \square \]

Corollary 1. $\dim_2(K_1 + \overline{K}_n) = |V(\overline{K}_n)|$.

Theorem 3. Let $G$ be a connected non-trivial graph and let $K_1 = \{v\}$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and $S$ is strictly 2-locating set of $G$ or $S = \{v\} \cup T$, where $T$ is a strictly 1-locating set in $G$.

Proof. Let $S \subseteq V(K_1 + G)$ be a 2-resolving set of $K_1 + G$. If $v \notin S$, then $S \subseteq V(G)$ is 2-locating set in $G$. Suppose there exists $y \in V(G)$ such that $|N_G(y) \cap S| > |S| - 2$. Then $r(v/S)$ and $r(y/S)$ differ in at most one position, contrary to our assumption that $S$ is a 2-resolving set in $K_1 + G$. Hence, $S$ is a strictly 2-locating set of $K_1 + G$. Next, suppose that $S = T \cup \{v\}$, where $T = V(G) \cap S$. Then $\emptyset \neq T \subseteq V(G)$. Thus, $T$ is a 2-locating set in $G$. Since $S$ is a 2-resolving set and $v \in S$, $T$ is strictly 1-locating set in $G$.

For the converse, let $x, y \in V(K_1 + G)$. First, assume that $v \notin S$ and $S$ is a strictly 2-locating set in $G$. Consider the following cases.

Case 1. $x, y \in S$

By Remark 3, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions, the $x^{th}$ and $y^{th}$ positions.

Case 2. $x, y \in V(G) \setminus S$

By Definition 3(i), $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the $z^{th}$ and $w^{th}$ positions, for some distinct vertices $z, w \in S$.

Case 3. $x \in S, y \in V(G) \setminus S$

By Definition 3(ii), there exists $z \in (N_G(x) \cap S) \setminus N_G(y)$ or $z \in (N_G(y) \cap S) \setminus N_G(x)$. Hence, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the $x^{th}$ and $z^{th}$ positions.

Case 4. $x = v, y \in V(G)$

By Definition 4, $\exists u, w \in S \setminus N_G(y), u \neq w$. Thus, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the $u^{th}$ and $w^{th}$ positions.

Next, suppose $S = \{v\} \cup T$ where $T$ is strictly 1-locating set in $G$. Consider the following cases.

Case 1. $x, y \in S$

By Remark 3, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions, the $x^{th}$ and $y^{th}$ positions.

Case 2. $x, y \in V(K_1 + G) \setminus S$

Then $x, y \in V(G) \setminus T$. By Definition 3(i), $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in at least 2 positions.

Case 3. $x = v, y \in V(G)$

By Definition 4, $\exists z \in T \setminus N_G(y)$. Thus, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ in the $x^{th}$
and $z^{th}$ positions.

**Case 4.** $x \in T$, $y \in V(G) \setminus T$.

Since $T$ is 2-locating set in $G$, $r_G(x/T)$ and $r_G(y/T)$ differ in at least 2 positions.

Hence, $r_{K_1+G}(x/S)$ and $r_{K_1+G}(y/S)$ differ also in at least 2 positions.

Therefore, $S$ is a 2-resolving set in $K_1 + G$.

The sets $\{u, u_1, u_3, u_4\}$ and $\{v, v_1, v_3, v_5\}$ are 2-resolving sets in the join $\langle u \rangle + P_5$ and $\langle v \rangle + C_6$, respectively, in Figure 3.

\[ \text{Figure 3: The join $\langle u \rangle + P_5$ with dim}_2(\langle u \rangle + P_5) = 4 \text{ and the join $\langle v \rangle + C_6$ with dim}_2(\langle v \rangle + C_6) = 4 \]

The next result follows immediately from Theorem 3.

**Corollary 2.** $\dim_2(K_1 + G) = \min \{ sln_2(G), sln_1(G) + 1 \}$.

**Example 11.**[8] For any integer $n \geq 6$, $\dim_2(F_{1,n}) = \left\lceil \frac{n+1}{2} \right\rceil = sln_2(P_n)$.

**Example 12.**[8] For any $n \geq 7$, $\dim_2(W_{1,n}) = \left\lceil \frac{n}{2} \right\rceil = sln_2(C_n)$.

**Theorem 4.** Let $G$ and $H$ be nontrivial connected graphs. A proper subset $S$ of $V(G+H)$ is a 2-resolving set in $G+H$ and only if $S_G = V(H) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in $G$ and $H$ respectively where $S_G$ or $S_H$ is strictly 2-locating set or $S_G$ and $S_H$ are strictly 1-locating sets.

**Proof.** Suppose $S$ is a proper subset of $V(G+H)$. Let $S$ be a 2-resolving set in $G+H$. Let $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$. Then $S = S_G \cup S_H$. Suppose $S_G = \emptyset$. Then $S = S_H$. Let $x, y \in V(G)$, $x \neq y$. Then $r_{G+H}(x/S) = r_{G+H}(y/S) = (1, \ldots, 1)$. A contradiction to the assumption of $S$. Thus, $S_G \neq \emptyset$. Similarly, $S_H \neq \emptyset$.

Next, suppose $S_G$ or $S_H$, say $S_G$ is not 2-locating set in $G$. Then there exist $x, y \in V(G)$, $x \neq y$ such that $r_G(x/S_G)$ and $r_G(y/S_G)$ differ in at most 1 position. Hence, $r_{G+H}(x/S)$ and $r_{G+H}(y/S)$ differ also in at most one position. Thus, $S$ is not 2-resolving set in $G+H$, contrary to our assumption. Therefore $S_G$ and $S_H$ are 2-locating sets in $G$ and $H$, respectively. Now, suppose that both $S_G$ and $S_H$ are not strictly 2-locating sets.

Then $|N_G(x) \cap S_G| > |S_G| - 2$, $\forall x \in V(G)$ and $|N_H(y) \cap S_H| > |S_H| - 2$, $\forall y \in V(H)$.

Hence either $N_G(x) \cap S_G = S_G$ or $\exists p \in S_G \setminus N_G(x)$ and either $N_H(y) \cap S_H = S_H$ or $\exists q \in S_H \setminus N_H(y)$. Since $S$ is a 2-locating set, $\exists p \in S_G \setminus N_G(x)$ and $\exists q \in S_H \setminus N_H(y)$. Thus, $S_G$ and $S_H$ are both strictly 1-locating sets.
For the converse, suppose that \(S_G\) and \(S_H\) are 2-locating sets in \(G\) and \(H\), respectively where \(S_G\) or \(S_H\) is strictly 2-locating set or \(S_G\) and \(S_H\) are both strictly 1-locating sets. Let \(x, y \in V(G + H)\) with \(x \neq y\). If \(x, y \in V(G)\), then \(r_G(x/S_G)\) and \(r_G(y/S_G)\) differ in at least 2 positions since \(S_G\) is a 2-locating set in \(G\). Hence, \(r_{G+H}(x/S)\) and \(r_{G+H}(y/S)\) also differ in at least 2 positions. Similarly, if \(x, y \in V(H)\), then \(r_{G+H}(x/S)\) and \(r_{G+H}(y/S)\) differ in at least 2 positions. Suppose that \(x \in V(G)\) and \(y \in V(H)\) and \(S_G\) is strictly 2-locating set. Then, \(\exists w, z \in S_G \setminus N_G(x)\). Then \(r_{G+H}(x/S)\) and \(r_{G+H}(y/S)\) differ in the \(z^{th}\) and \(w^{th}\) positions. Therefore, \(S\) is a 2-resolving set in \(G + H\).

**Corollary 3.** Let \(G\) and \(H\) be connected nontrivial graphs. Then,

\[
\dim_2(G + H) = \min \{ \text{sln}_2(G) + \text{ln}_2(H), \text{ln}_2(G) + \text{sln}_2(H), \text{sln}_1(G) + \text{sln}_1(H) \}
\]

**Proof.** Let \(S\) be a minimum 2-resolving set of \(G + H\). Let \(S_G = V(G) \cap S\) and \(S_H = V(H) \cap S\). By Theorem 4, \(S_G\) and \(S_H\) are 2-locating sets in \(G\) and \(H\), respectively where \(S_G\) or \(S_H\) is strictly 2-locating set or \(S_G\) and \(S_H\) are both strictly 1-locating sets. If \(S_G\) is strictly 2-locating set in \(G\), then \(\text{sln}_2(G) + \text{ln}_2(H) \leq |S_G| + |S_H| = |S| = \dim_2(G + H)\). If \(S_H\) is strictly 2-locating set in \(H\), then \(\text{sln}_2(H) + \text{ln}_2(G) \leq |S_H| + |S_G| = |S| = \dim_2(G + H)\). Thus, \(\dim_2(G + H) \geq \min \{ \text{sln}_2(G) + \text{ln}_2(H), \text{ln}_2(G) + \text{sln}_2(H), \text{sln}_1(G) + \text{sln}_1(H) \}\). Next suppose that \(\text{sln}_1(G) + \text{sln}_1(H) \leq \text{sln}_2(G) + \text{ln}_2(H)\) and \(\text{sln}_1(G) + \text{sln}_1(H) \leq \text{ln}_2(G) + \text{sln}_2(H)\). Let \(S_G\) be a minimum strictly 1-locating set in \(G\) and \(S_H\) be a minimum strictly 1-locating set in \(H\). Then \(S = S_G \cup S_H\) is a 2-resolving set in \(G + H\), by Theorem 4. Hence \(\dim_2(G + H) \leq |S| = |S_G| + |S_H| = \text{sln}_1(G) + \text{sln}_1(H)\). Therefore, \(\dim_2(G + H) = \text{sln}_1(G) + \text{sln}_1(H)\). Similarly, if \(\text{sln}_2(G) + \text{ln}_2(H) \leq \text{sln}_1(G) + \text{sln}_1(H)\) and \(\text{sln}_2(G) + \text{ln}_2(H) \leq \text{ln}_2(G) + \text{sln}_2(H)\), then \(\dim_2(G + H) \leq \text{sln}_2(G) + \text{ln}_2(H)\). Also, if \(\text{ln}_2(G) + \text{sln}_2(H) \leq \text{sln}_1(G) + \text{ln}_2(H)\) and \(\text{ln}_2(G) + \text{sln}_2(H) \leq \text{ln}_2(G) + \text{sln}_2(H)\), then \(\dim_2(G + H) \leq \text{sln}_2(G) + \text{ln}_2(H)\). Therefore, \(\dim_2(G + H) = \min \{ \text{sln}_2(G) + \text{ln}_2(H), \text{ln}_2(G) + \text{sln}_2(H), \text{sln}_1(G) + \text{sln}_1(H) \}\).

**Example 13.** For any \(n, m \geq 4\),

\[
\dim_2(P_n + P_m) = \begin{cases} 
\left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) + \left(\left\lceil \frac{m}{2} \right\rceil + 1 \right) & \text{if } n, m \text{ even} \\
\left(\left\lceil \frac{n}{2} \right\rceil + 1 \right) + \left(\left\lfloor \frac{m}{2} \right\rfloor \right) & \text{if } n \text{ is even, } m \text{ is odd} \\
\left\lceil \frac{n}{2} \right\rceil + \left(\left\lfloor \frac{m}{2} \right\rfloor + 1 \right) & \text{if } n \text{ is odd, } m \text{ is even} \\
\left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{m}{2} \right\rfloor & \text{if } n, m \text{ odd}
\end{cases}
\]

In particular, for \(n = 2, 3\) and \(m = 2, 3\),

\[\dim_2(P_n + P_m) = n + m\]
4. 2-Resolving Sets in the Corona of Graphs

**Definition 5.** [2] The corona \( G \circ H \) of two graphs \( G \) and \( H \) is the graph obtained by taking one copy of \( G \) of order \( n \) and \( n \) copies of \( H \), and then joining the \( i \)th vertex of \( G \) to every vertex in the \( i \)th copy of \( H \). For every \( v \in V(G) \), denote by \( H^v \) the copy of \( H \) whose vertices are attached one by one to the vertex \( v \). Subsequently, denote by \( v + H^v \) the subgraph of the corona \( G \circ H \) corresponding to the join \( \langle \{v\} \rangle + H^v \), \( v \in V(G) \).

The sets \( \{u_1, u_2, v_1, v_2, w_1, w_2\} \) and \( \{a_1, a_3, b_1, b_3, c_1, c_3, d_1, d_3\} \) are 2-resolving sets in the coronas \( P_3 \circ P_2 \) and \( C_4 \circ P_3 \), respectively, in Figure 4.

**Remark 6.** Let \( v \in V(G) \). For every \( x, y \in V(H^v) \), \( d_{G \circ H}(x, w) = d_{G \circ H}(y, w) \) and \( d_{G \circ H}(v, w) + 1 = d_{G \circ H}(x, w) \) for every \( w \in V(G \circ H) \setminus V(H^v) \).

**Remark 7.** Let \( G \) and \( H \) be non-trivial connectd graphs, \( C \subseteq V(G \circ H) \) and \( S_v = V(H^v) \cap C \) where \( v \in V(G) \). For each \( x \in V(H^v) \setminus S_v \) and \( z \in S_v \),

\[
d_{G \circ H}(x, z) = \begin{cases} 1 & \text{if } z \in N_{H^v}(x) \\ 2 & \text{otherwise} \end{cases}
\]

**Theorem 5.** Let \( G \) and \( H \) be nontrivial connected graphs. A proper subset \( S \) of \( V(G \circ H) \) is a 2-resolving set of \( G \circ H \) if and only if \( S = A \cup B \), where \( A \subseteq V(G) \) and

\[
B = \bigcup \{ S_v : S_v \text{ is a 2-resolving set of } H^v, \forall v \in V(G) \}.
\]

**Proof.** Suppose \( S \) is a 2-resolving set in \( G \circ H \). Let \( A = V(G) \cap C \) and \( S_v = S \cap V(H^v) \) for all \( v \in V(G) \). Then \( S = A \cup \bigcup _{v \in V(G)} S_v \) where \( A \subseteq V(G) \) and \( S_v \subseteq V(H^v) \). Suppose \( S_v = \emptyset \) for some \( v \in V(G) \). Let \( x, y \in V(H^v) \). Then \( r_{G \circ H}(x/S) = r_{G \circ H}(y/S) \) which is a contradiction to the assumption of \( S \). Thus \( S_v \neq \emptyset \). Now, we claim that \( S_v \) is a 2-resolving set in \( H^v \) for each \( v \in V(G) \). Let \( p, q \in V(H^v) \) where \( p \neq q \). Since \( S \) is a 2-resolving set in \( G \circ H \), \( r_{G \circ H}(p/S) \) and \( r_{G \circ H}(q/S) \) differ in at least 2 positions. By Remark 6, \( r_{H^v}(p/S_v) \) and \( r_{H^v}(q/S_v) \) must differ in at least 2 positions. Thus \( S_v \) is a 2-resolving set in \( H^v \).
Conversely, let $S = A \cup \left( \bigcup_{v \in V(G)} S_v \right)$ where $A \subseteq V(G)$ and $S_v \subseteq V(H^v)$ satisfying the given conditions. Let $x, y \in V(G \circ H)$ with $x \neq y$ and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$.

**Case 1.** $u = v$

**Subcase 1.1** $x, y \in V(H^u)$

Since $S_v$ is a 2-resolving set, $r_{H^v}(x/S_v)$ and $r_{H^v}(y/S_v)$ differ in at least 2 positions.

By Remark 6, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Subcase 1.2** $x = v$ and $y \in V(H^v)$

Since $G$ is nontrivial and connected, $\exists w \in N_G(v)$ and $|S_w| \geq 2$. By Remark 6, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Case 2.** $u \neq v$

**Subcase 2.1** $x \in V(H^u)$, $y \in V(H^v)$

Note that $r_{G \circ H}(x/S_v)$ has components greater than or equal to 3 and $r_{G \circ H}(y/S_v)$ has components less than or equal to 2. Since $|S_v| \geq 2$, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

**Subcase 2.2** $x = u$, $y \in V(v + H^v)$

Since $|S_u| \geq 2$, $r_{G \circ H}(x/S_u)$ and $r_{G \circ H}(y/S_u)$ differ in at least 2 positions. Hence, $r_{G \circ H}(x/S)$ and $r_{G \circ H}(y/S)$ differ in at least 2 positions.

Therefore, in any case, $S$ is a 2-resolving set in $G \circ H$.

**Corollary 4.** Let $G$ and $H$ be nontrivial connected graphs, where $|V(G)| = n$. Then $\dim_2(G \circ H) = n \cdot \dim_2(H)$.

**Proof.** Let $S$ be a minimum 2-resolving set of $G \circ H$. Then by Theorem 5, $S = A \cup B$, where $A \subseteq V(G)$ and $B = \bigcup S_v$, $v \in V(G)$ and $S_v$ is a 2-resolving set in $H$. Hence,

$$\dim_2(G \circ H) = |S| = |A| + |B| \\
\geq |A| + |V(G)| \cdot \dim_2(H) \\
= |A| + n \cdot \dim_2(H) \\
\geq n \cdot \dim_2(H).$$

Now, let $C$ be a minimum 2-resolving set in $H$. For each $v \in V(G)$, choose $C_v \subseteq V(H^v)$ with $(C_v) \cong (C)$. Then $D = \bigcup_{v \in V(G)} C_v$ is a 2-resolving set in $G \circ H$ by Theorem 5. Hence,

$$\dim_2(G \circ H) \leq |D| = \left| \bigcup_{v \in V(G)} C_v \right| = n \cdot |C_v| = n \cdot |C| = n \cdot \dim_2(H).$$

Therefore, $\dim_2(G \circ H) = n \cdot \dim_2(H)$.

**Example 14.** For any integer $n \geq 2$ and $m \geq 5$,

$$\dim_2(G \circ C_m) = \begin{cases} 
 n \left\lfloor \frac{m}{2} \right\rfloor, & \text{if } m \text{ is odd} \\
 n \left\lceil \frac{m}{2} \right\rceil, & \text{if } m \text{ is even}
\end{cases}$$
Example 15. For any integer $n, m \geq 2$,

$$\dim_2(G \circ P_m) = \begin{cases} n \left( \left\lceil \frac{m}{2} \right\rceil \right), & \text{if } m \text{ is odd} \\ n \left( \left\lfloor \frac{m}{2} \right\rfloor \right) + 1, & \text{if } m \text{ is even} \end{cases}$$

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References


