Global Hop Domination Numbers of Graphs

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Abstract. A set $S \subseteq V(G)$ is a hop dominating set of $G$ if for each $v \in V(G) \setminus S$, there exists $w \in S$ such that $d_G(v, w) = 2$. It is a global hop dominating set of $G$ if it is a hop dominating set of both $G$ and the complement $\overline{G}$ of $G$. The minimum cardinality of a global hop dominating set of $G$, denoted by $\gamma_{gh}(G)$, is called the global hop domination number of $G$. In this paper, we study the concept of global hop domination in graphs resulting from some binary operations.

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1. Introduction

Domination is a well-studied topic in Graph Theory. From the standard concept, many other variations of domination have been investigated by researchers. Connected, total, independent, and global domination are among the numerous well-known variants of the standard domination concept. Other variants may be found in the two books authored by Haynes et al. (see [5] and [6]).

Recently, Natarajan and Ayyaswamy [10] introduced and studied the concept of hop domination in a graph. In another study, Ayyaswamy et al. [1] investigated the same concept and gave bounds of the hop domination number of some graphs. Henning and Rad [7] also studied the concept and answered a question posed by Ayyaswamy and Natarajan in [10]. They showed that the hop dominating set problem is NP-complete for planar bipartite graphs and planar chordal graphs. Hop domination and some of its variants are studied in [3], [8], [9], and [11]. In this paper, we study another variation of hop domination called global hop domination. This is obviously the analogue to global domination studied in [2] and [4].

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Let $G = (V(G), E(G))$ be a simple graph. The distance between two vertices $u$ and $v$ of $G$, denoted by $d_G(u, v)$, is equal to the length of a shortest path connecting $u$ and $v$. Any path connecting $u$ and $v$ of length $d_G(u, v)$ is called a $u$-$v$ geodesic. The open neighbourhood of a vertex $v$ of $G$ is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its closed neighbourhood is the set $N_G[v] = N_G(v) \cup \{v\}$. The open neighbourhood of a subset $S$ of $V(G)$ is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and its closed neighbourhood is the set $N_G[S] = N_G(S) \cup S$. The degree of $v$, denoted by $deg_G(v)$, is equal to $|N_G(v)|$. The minimum degree of $G$ is $\delta(G) = \min\{deg_G(v) : v \in V(G)\}$ and its maximum degree is $\Delta(G) = \max\{deg_G(v) : v \in V(G)\}$. The open hop neighbourhood of vertex $v$ of $G$ is the set $N_G(v, 2) = \{w \in V(G) : d_G(v, w) = 2\}$. A set $S \subseteq V(G)$ is a dominating set (resp. total dominating set) of $G$ if $N_G[S] = V(G)$ (resp. $N_G(S) = V(G)$). The smallest cardinality of a dominating (resp. total dominating) set of $G$, denoted by $\gamma(G)$ (resp. $\gamma_t(G)$), is called the domination number (resp. total domination number) of $G$. A dominating (resp. total dominating) set of $G$ with cardinality $\gamma(G)$ (resp. $\gamma_t(G)$), is called a $\gamma$-set (resp. $\gamma_t$-set) of $G$. It should be noted that only graphs without isolated vertices admit total dominating sets.

A set $S \subseteq V(G)$ is a hop dominating set of $G$ if for each $x \in V(G) \setminus S$, there exists $z \in S$ such that $d_G(x, z) = 2$. The smallest cardinality of a hop dominating set of $G$, denoted by $\gamma_h(G)$, is called the hop domination number of $G$. A hop dominating set of $G$ with cardinality $\gamma_h(G)$ is called a $\gamma_h$-set of $G$. A set $S \subseteq V(G)$ is a global hop dominating set of $G$ if it is a hop dominating set of $G$ and $\overline{G}$. The smallest cardinality of a global hop dominating set of $G$, denoted by $\gamma_{gh}(G)$, is called the global hop domination number of $G$. A global hop dominating set of $G$ with cardinality $\gamma_{gh}(G)$ is called a $\gamma_{gh}$-set of $G$.

A set $D \subseteq V(G)$ is a pointwise non-dominating set of $G$ if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $v \notin N_G(u)$. The smallest cardinality of a pointwise non-dominating set of $G$, denoted by $\text{pnd}(G)$, is called the pointwise non-dominating number of $G$. A dominating set $S$ which is also a pointwise non-dominating set of $G$ is called a dominating pointwise non-dominating set of $G$. The smallest cardinality of a dominating pointwise non-dominating set of $G$ will be denoted by $\gamma_{\text{pnd}}(G)$. Any pointwise non-dominating (resp. dominating pointwise non-dominating) set of $G$ with cardinality $\text{pnd}(G)$ (resp. $\gamma_{\text{pnd}}(G)$), is called a $\text{pnd}$-set (resp. $\gamma_{\text{pnd}}$-set) of $G$. These concepts and parameters have been defined and used in [3] and [9].

2. Results

It is worth mentioning here that every graph $G$ admits a global hop dominating set. Indeed, the vertex set $V(G)$ of $G$ is a global hop dominating set. Further, we have

**Remark 1.** $1 \leq \gamma_{gh}(G) \leq |V(G)|$ for any graph $G$. Moreover, $\gamma_{gh}(G) = 1$ if and only if $G = K_1$.

**Theorem 1.** Let $G$ be a non-trivial graph. Then $\gamma_{gh}(G) = 2$ if and only if there exist distinct vertices $x$ and $y$ of $G$ satisfying the following conditions:
(i) \( N_G(x, 2) \cap N_G(y, 2) = \emptyset \) and \( V(G) \setminus \{x, y\} = N_G(x, 2) \cup N_G(y, 2) \);
(ii) \( N_G(x, 2) = N_G(y) \setminus \{x\} \) and \( N_G(y, 2) = N_G(x) \setminus \{y\} \); and

(iii) if \( xy \in E(G) \), then \( N_G(x) \setminus N_G(w) \neq \emptyset \) for each \( w \in N_G(x, 2) \) and \( N_G(y) \setminus N_G(v) \neq \emptyset \) for each \( v \in N_G(y, 2) \).

Proof. Suppose \( \gamma_{gh}(G) = 2 \). Let \( S = \{x, y\} \) be \( \gamma_{gh} \)-set of \( G \). Suppose there exists \( z \in N_G(x, 2) \setminus N_G(y, 2) \). Then \( xz, yz \in E(G) \). This implies that \( d_{\overline{G}}(x, z) \neq 2 \) and \( d_{\overline{G}}(y, z) \neq 2 \). Hence, \( S \) is not hop dominating set of \( \overline{G} \), a contradiction. Thus, \( N_G(x, 2) \cap N_G(y, 2) = \emptyset \).

Further, \( V(G) \setminus \{x, y\} = N_G(x, 2) \cup N_G(y, 2) \) because \( S \) is a hop dominating set of \( G \). This shows that (i) holds.

Now let \( z \in N_G(x, 2) \). Then \( z \notin S \) and \( xz \in V(\overline{G}) \). Since \( S \) is hop dominating set of \( \overline{G} \), it follows that \( z \in N_{\overline{G}}(y, 2) \). This implies that \( z \notin N_G(y) \setminus \{x\} \). On the other hand, if \( u \in N_G(y) \setminus \{x\} \), then \( u \in N_G(x, 2) \) since \( S \) is a hop dominating set of \( G \). Therefore, \( N_G(x, 2) = N_G(y) \setminus \{x\} \). Similarly, \( N_G(y, 2) = N_G(x) \setminus \{y\} \), showing that (ii) holds.

Next, suppose that \( xy \in E(G) \) and let \( w \in N_G(x, 2) \). Then \( w \notin S \) and \( xw \in V(\overline{G}) \).

Since \( S \) is a hop dominating set of \( \overline{G} \), \( w \notin N_{\overline{G}}(y, 2) \). Hence, there exists \( z \in V(G) \setminus S \) such that \( z \notin N_{\overline{G}}(w) \cap N_{\overline{G}}(y) \), i.e., \( N_G(x, 2) \cap N_G(w) \neq \emptyset \). Similarly, \( N_G(y) \setminus N_G(v) \neq \emptyset \) for each \( v \in N_G(y, 2) \), showing that (iii) holds.

Conversely, suppose that there exist distinct vertices \( x \) and \( y \) of \( G \) satisfying conditions (i), (ii), and (iii). Let \( S = \{x, y\} \). By (i), \( S \) is a hop dominating set of \( G \). Let \( v \in V(\overline{G}) \setminus S \). Assume, without loss of generality, that \( v \notin N_G(x, 2) \). Then \( v \notin N_G(y) \setminus \{x\} \) by (ii). Suppose \( xy \notin E(G) \). Then \( xy, xv \in E(\overline{G}) \). Thus, \( d_{\overline{G}}(y, v) = 2 \). Next, suppose that \( xy \in E(G) \). Then by (iii), there exists \( z \in N_G(x) \setminus N_G(v) \). Hence, \( z \in N_{\overline{G}}(v) \cap N_{\overline{G}}(y) \), i.e., \( d_{\overline{G}}(y, v) = 2 \). Therefore, \( S \) is a global hop dominating set of \( G \). Accordingly, \( \gamma_{gh}(G) = 2 \). \( \square \)

**Theorem 2.** Let \( G \) be a graph of order \( n \geq 2 \). Then \( \gamma_{gh}(G) = n \) if and only if one of the following statements holds:

(i) Every component of \( G \) is complete.
(ii) For each \( v \in V(G) \), \( V(G) \setminus N_G(v) \) is an independent set and \( N_G(v) = N_G(a) \) for each \( a \in V(G) \setminus N_G(v) \).

Proof. Suppose \( \gamma_{gh}(G) = n \). Suppose first that \( G \) is disconnected and suppose that \( G \) has a component \( C \) which is not complete. Then there exist distinct vertices \( x, y \in V(C) \) such that \( d_G(x, y) = d_C(x, y) = 2 \). Let \( S = V(G) \setminus \{x\} \). Then \( S \) is a hop dominating set of \( G \). Let \( z \in C \) such that \( [x, z, y] \) is an \( x-y \) geodesic in \( G \). Let \( C' \) be a component of \( G \) with \( C' \neq C \) and pick any \( w \in C' \). Then \( [x, w, z] \) is an \( x-z \) geodesic in \( \overline{G} \). It follows that \( d_{\overline{G}}(x, z) = 2 \). Thus, \( S \) is a hop dominating set of \( \overline{G} \), showing that \( S \) is a global hop dominating set of \( G \). Therefore, \( \gamma_{gh}(G) \leq |S| = n - 1 \), a contradiction. Accordingly, every component of \( G \) is complete.

Next, suppose that \( G \) is connected. Suppose further that \( \overline{G} \) is connected. Then, clearly, \( G \neq K_n \). Let \( u, v \in V(G) \) such that \( d_G(u, v) = 2 \) and let \( [u, p, v] \) be a \( u-v \)
Let $S^* = V(G) \setminus \{u\}$ be a hop dominating set of $G$. Since $u \notin E(G)$, it follows that $d_{\overline{G}}(u, p) \geq 2$. It follows that there exists $q \in S$ such that $d_{\overline{G}}(u, q) = 2$. This shows that $S^*$ is hop dominating set of $G$. Thus, $S^*$ is a global hop dominating set of $G$ and $\gamma_{gh}(G) \leq |S^*| = n - 1$, a contradiction. Therefore $\overline{G}$ is disconnected. Since $\gamma_{gh}(\overline{G}) = \gamma_{gh}(G) = n$, this would imply that every component of $\overline{G}$ is complete (as in the first case applied to $\overline{G}$). Let $v \in V(G) = V(\overline{G})$ and suppose there exist distinct vertices $a, b \in V(G) \setminus N_G(v)$ such that $ab \in E(G)$. Then $[a, v, b]$ is an $a-b$ geodesic in $\overline{G}$, implying that $S_a = V(G) \setminus \{a\}$ is a hop dominating set of $\overline{G}$. Now, since $a \in V(G) \setminus N_G(v)$, it follows that $d_G(a, v) \geq 2$. This implies that there exists $w \in S_a$ such that $d_G(a, w) = 2$, showing that $S_a$ is also a hop dominating set of $G$. Hence, $\gamma_{gh}(G) \leq |S_a| = n - 1$, a contradiction. Therefore, $V(G) \setminus N_G(v)$ is an independent set. Let $a \in V(G) \setminus N_G(v)$. Let $C_v$ be the component of $\overline{G}$ with $v \in C_v$. Since $a \in N_{\overline{G}}(v)$ and $C_v$ is complete, $N_G(a) = N_G(v)$, that is, $az \in E(G)$ for every $z \in N_G(v)$. This shows that (ii) holds.

For the converse, suppose first that (i) holds. Then, clearly, $S = V(G)$ is the only hop dominating set of $G$. It follows that $S$ is the only global hop dominating set of $G$. Thus, $\gamma_{gh}(G) = n$. Next, suppose that (ii) holds. Then every component of $\overline{G}$ is complete. Since $V(G) = V(\overline{G})$ is the only hop dominating set of $\overline{G}$, it follows that $V(G)$ is the only global hop dominating set of $G$. Therefore, $\gamma_{gh}(G) = n$.

The next result is a consequence of Theorem 2.

**Corollary 1.** $\gamma_{gh}(K_n) = \gamma_{gh}(K_{1,n-1}) = n$ for all integer $n \geq 2$.

A set $S \subseteq V(G)$ is a pairwise non-dominating set of $G$ if for each $v \in V(G) \setminus S$, there exists vertex $w \in S \cap N_G(v)$ such that $N_G(\{w, v\}) \neq V(G)$. A set $S \subseteq V(G)$ is a pairwise and pointwise non-dominating (ppnd) set of $G$ if it is both a pairwise non-dominating and pointwise non-dominating set of $G$. The minimum cardinality of a ppnd set of $G$ is denoted by $\gamma_{ppnd}(G)$. Any pairwise and pointwise non-dominating set of $G$ with cardinality equal to $\gamma_{ppnd}(G)$ is called a $\gamma_{ppnd}$-set of $G$.

**Remark 2.** A pairwise non-dominating set of $G$ is a dominating set of $G$.

**Theorem 3.** Let $G$ be any graph of order $n$. Then $1 \leq \gamma_{ppnd}(G) \leq n$. Moreover,

(i) $\gamma_{ppnd}(G) = 1$ if and only if $G = K_1$,

(ii) $\gamma_{ppnd}(G) = 2$ if and only if one of the following statements holds:

(a) $G = K_2$

(b) $G = \overline{K}_2$

(c) There exist non-adjacent vertices $x, y \in V(G)$ such that $N_G(x) \cap N_G(y) = \emptyset$ and $N_G[x] \cup N_G[y] = V(G)$.

(d) There exist adjacent vertices $x, y \in V(G)$ such that $N_G(x) \cap N_G(y) = \emptyset$, $N_G(x) \cup N_G(y) = V(G)$, and for each $v \in N_G(x) \setminus \{y\}$ and $w \in N_G(y) \setminus \{x\}$, there exist $p \in N_G(y) \setminus N_G(v)$ and $q \in N_G(x) \setminus N_G(w)$.
$(iii)$ $\gamma_{ppnd}(G) = n$ if and only if $G = K_n$ or $G$ is connected such that $N_G(\{u,v\}) = V(G)$ for each pair of adjacent vertices $u,v \in V(G)$.

**Proof.** Clearly, by definition, a pairwise and pointwise non-dominating set of $G$ is nonempty. Thus, $\gamma_{ppnd}(G) \geq 1$. Also, since $V(G)$ is a pairwise and pointwise non-dominating set of $G$, it follows that $\gamma_{ppnd}(G) \leq n$.

(i) Next, suppose that $\gamma_{ppnd}(G) = 1$, say $S = \{v\}$ is a $\gamma_{ppnd}$-set of $G$. If such a vertex outside $S$ exists, then this would require two distinct vertices from $S$ to satisfy the property of $S$. This forces us to conclude that $G = K_1$. Further, since $\gamma_{ppnd}(K_1) = 1$, (i) holds.

(ii) Suppose now that $\gamma_{ppnd}(G) = 2$, say $S = \{x,y\}$ is a $\gamma_{ppnd}$-set of $G$. If $n = 2$, then $G = K_2$ or $G = \overline{K}_2$. Suppose $n \geq 3$ and assume first that $xy \notin E(G)$. Since $S$ is a $ppnd$ set of $G$, $N_G(x) \cap N_G(y) = \emptyset$ and $N_G[x] \cup N_G[y] = V(G)$. Hence, (e) holds. Suppose $xy \in E(G)$. Again, since $S$ a $ppnd$ set of $G$, $N_G(x) \cap N_G(y) = \emptyset$ and $N_G(x) \cup N_G(y) = V(G)$. Let $v \in N_G(x) \setminus \{y\}$. Since $N_G(\{x,y\}) \neq V(G)$, there exists $p \in V(G) \setminus N_G(\{x,v\})$. Since $N_G(x) \cap N_G(y) = \emptyset$, it follows that $p \notin N_G(y) \setminus N_G(v)$. Similarly, for each $w \in N_G(y) \setminus \{x\}$, there exists $q \in N_G(x) \setminus N_G(w)$, showing that (d) holds.

For the converse, suppose first that $G = K_2$ or $G = \overline{K}_2$. Then, clearly, $\gamma_{ppnd}(G) = 2$. Next, suppose that (c) holds. Let $S = \{x,y\}$ and let $v \in V(G) \setminus S$. By assumption, we may assume that $v \in N_G(x) \setminus N_G(y)$. Since $y \in V(G) \setminus N_G(\{x,v\})$, $N_G(\{x,v\}) \neq V(G)$. Thus, $S$ is a $ppnd$ set of $G$. Since $G \neq K_1$, it follows that $S$ is a $\gamma_{ppnd}$-set, i.e., $\gamma_{ppnd}(G) = |S| = 2$.

Finally, suppose that (d) holds. Let $S' = \{x,y\}$ and let $v \in V(G) \setminus S$. Assume, without loss of generality, that $v \in N_G(x)$. By assumption, there exists $p \in N_G(y) \setminus N_G(v)$. This implies that $p \notin N_G(\{x,v\})$. Therefore, $S$ is a $\gamma_{ppnd}$-set of $G$, implying that $\gamma_{ppnd}(G) = 2$. This proves statement (ii).

(iii) Suppose $\gamma_{ppnd}(G) = n$. Suppose first that $G$ is disconnected. Suppose further that $G \neq K_n$. Then $G$ has a non-trivial component $C$. Hence, there exist distinct vertices $x,y \in V(C)$ such that $xy \in E(G)$. Let $S_x = V(G) \setminus \{x\}$. Then $y \in S \cap N_G(x)$. Since $G$ is disconnected, $N_G(x,y) \neq V(G)$ and there exists $w \in S \setminus N_G(x)$. Hence, $S$ is a $ppnd$ set of $G$ and $\gamma_{ppnd}(G) \leq |S| = n - 1$, a contradiction. Therefore, $G = K_n$.

Next, suppose that $G$ is connected. Suppose there exist distinct adjacent vertices $u,v \in V(G)$ such that $N_G(\{u,v\}) \neq V(G)$, say $w \in V(G) \setminus N_G(\{u,v\})$. Let $S_u = V(G) \setminus \{u\}$. Then $v,w \in S, uw \notin E(G), uv \in E(G)$, and $N_G(\{u,v\}) \neq V(G)$. This implies that $S$ is a pairwise and pointwise non-dominating set of $G$. Hence, $\gamma_{ppnd}(G) \leq |S| = n - 1$, a contradiction. Therefore, $N_G(\{u,v\}) = V(G)$ for each pair of adjacent vertices $u,v \in V(G)$.

For the converse, suppose first that $G = K_n$. Then, clearly, $S = V(G)$ is the only pairwise and pointwise non-dominating set of $G$. Thus, $\gamma_{ppnd}(G) = n$. Next, suppose that $G$ is connected and satisfies the condition that $N_G(\{u,v\}) = V(G)$ for each pair of adjacent vertices $u,v \in V(G)$. Let $S$ be a $\gamma_{ppnd}$-set and suppose that there exists $w \in V(G) \setminus S$. Then there exists $q \in S \cap N_G(w)$ such that $N_G(\{q,w\}) \neq V(G)$, contrary to our assumption. Therefore, $S = V(G)$ and $\gamma_{ppnd}(G) = n$. \qed
Corollary 2. Let $G$ and $H$ be any two graphs. Then $\gamma_{gh}(G+H) = \gamma_{ppnd}(G) + \gamma_{ppnd}(H)$. In particular,

(i) $\gamma_{gh}(K_n + H) = n + \gamma_{ppnd}(H)$ for all integer $n \geq 1$, and

(ii) $\gamma_{gh}(K_{m,n}) = m + n$ for all positive integers $m$ and $n$.

The next result is immediate from Theorem 4 and Theorem 3(iii).

Theorem 5. Let $G$ be a connected non-trivial graph and let $H$ be any graph. A set $C \subseteq V(G \circ H)$ is a global hop dominating set of $G \circ H$ if and only if $C = A \cup (\bigcup_{v \in V(G)} S_v)$, where $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfy the following properties:

(i) For each $w \in V(G) \setminus A$, there exists $x_w \in A$ with $d_G(w, x_w) = 2$ or there exists $y \in V(G) \cap N_G(w)$ with $S_y \neq \emptyset$.

(ii) $S_v$ is a dominating set of $H^v$ for each $v \in V(G) \setminus A$.

(iii) $S_v$ is a pointwise non-dominating set of $H^v$ for each $v \in A \setminus N_G(A)$.

(iv) $S_v$ is a dominating pointwise non-dominating set of $H^v$ for each $v \in V(G) \setminus N_G(A)$. 

Proof. Suppose $C$ is a global hop dominating set of $G \circ H$ and let $A = C \cap V(G)$. Let $S_v = C \cap V(H^v)$ for each $v \in V(G)$. Then $A \subseteq V(G)$, $S_v \subseteq V(H^v)$ for each $v \in V(G)$, and $C = A \cup (\cup_{v \in V(G)} S_v)$. Now, since $C$ is a hop dominating set of $G$, (i) holds. Next, let $v \in V(G)$ and consider the following cases:

Case 1: $v \in N_G(A) \backslash A$

Let $x \in V(H^v) \backslash S_v$. Since $C$ is hop dominating set of $G \circ H$, there exists $y \in C$ such that $d_{G \circ H}(x, y) = 2$. Since $v \notin A$ and $V(G \circ H) \setminus V(v + H^v) \subseteq N_{G \circ H}(x)$, it follows that $y \in S_v$. Thus, $y \in S_v \cap N_{H^v}(x)$, showing that $S_v$ is a dominating set of $H^v$. Therefore, (ii) holds.

Case 2: $v \in A \setminus N_G(A)$

Let $w \in A \setminus N_G(A)$ and let $q \in V(H^v) \setminus S_v$. Since $C$ is a hop dominating set of $G \circ H$, there exists $u \in C$ such that $d_{G \circ H}(q, u) = 2$. By assumption, $u \notin A$. Thus, $u \in S_v$ and $qu \notin E(H^v)$. Therefore, $S_v$ is a pointwise non-dominating set of $H^v$, showing that (iii) holds.

Case 3: $v \in V(G) \setminus N_G[A]$

Since $v \notin A$ and $C$ is a hop dominating set of $G$, similar arguments in Case 1 will show that $S_v$ is a dominating set of $H^v$. Further, since $v \notin N_G(A)$, the arguments in Case 2 can be used to show that $S_v$ is a pointwise non-dominating set of $H^v$, showing that (iv) holds.

For the converse, suppose that $C$ has the given form and satisfies properties (i), (ii), (iii), and (iv). Next, let $z \in V(G \circ H) \setminus C = V(G \circ H) \setminus C$ and let $v \in V(G)$ such that $z \in V(v + H^v)$. Consider the following cases:

Case 1. $z = v$

Then there exists $h \in C$ such that $d_{G \circ H}(z, h) = 2$, by (i). Now, from the assumption that (ii) and (iv) hold, it follows that $S_z = \emptyset$. Pick any $p \in S_z$ and $y \in V(H^w)$, where $w \in V(G) \cap N_G(z)$. Then $zy, yp \in E(G \circ H)$; hence, $d_{G \circ H}(z, p) = 2$.

Case 2. $z \neq v$

Then $z \in V(H^v) \setminus S_v$. If $v \notin N_G(A)$, then $d_{G \circ H}(z, a) = 2$ for $a \in A \cap N_G(v)$. If $v \notin N_G(A)$, then there exists $b \in S_v \subset C$ such that $d_{G \circ H}(z, b) = 2$ by (iii) and (iv).

Next, suppose first that $v \in A$. Pick any $w \in V(G) \setminus \{v\}$ and let $p \in V(H^w)$. Then $p \in N_{G \circ H}(z) \cap N_{G \circ H}(v)$. Thus, $d_{G \circ H}(z, v) = 2$. Suppose now that $v \notin A$. By (ii) and (iv), $S_v$ is a dominating set of $H^v$. It follows that there exists $q \in S_v \cap N_{H^v}(z)$. Pick any $u \in V(G) \setminus \{v\}$. Then $u \in N_{G \circ H}(z) \cap N_{G \circ H}(q)$. Hence, there exists $q \in C$ such that $d_{G \circ H}(z, q) = 2$.

Accordingly, $C$ is a hop dominating set of $G \circ H$ and $G \circ H$, showing that $C$ is a global hop dominating set of $G \circ H$.

Corollary 3. Let $G$ be a connected non-trivial graph and let $H$ be any graph. Then $\gamma_{gh}(G \circ H) = |V(G)|$.

Proof. Let $A = V(G)$ and set $S_v = \emptyset$ for each $v \in V(G)$. Then $C = A = A \cup (\cup_{v \in V(G)} S_v)$ is a global hop dominating set of $G$ by Theorem 5. Hence, $\gamma_{gh}(G \circ H) \leq |C| = |V(G)|$. 


Next, let $C_0$ be a $\gamma_{gh}$-set of $G \circ H$. Then $C_0 = A_0 \cup (\cup_{v \in V(G)} R_v)$, where $A_0 \subseteq V(G)$ and $R_v \subseteq V(H^v)$ for each $v \in V(G)$ and satisfy conditions (i), (ii), and (iv) of Theorem 5. Since $C_0$ is a $\gamma_{gh}$-set of $G \circ H$, it follows that $R_v = \emptyset$ for all $v \in D_1 = A_0 \cap N_G(A_0)$. From conditions (ii), (iii), and (iv), we find that $|R_v| \geq 1$ for each $v \in D_2 = V(G) \setminus D_1$. Thus, $\gamma_{gh}(G \circ H) = |C_0| = |A_0| + \sum_{v \in D_2} |R_v| \geq |A_0| + |D_2| = |V(G)| + (|A_0| - |D_1|) \geq |V(G)|$. Therefore, $\gamma_{gh}(G \circ H) = |V(G)|$. \(\square\)

The lexicographic product of graphs $G$ and $H$, denoted by $G[H]$, is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ such that $(v, a)(u, b) \in E(G[H])$ if and only if either $uv \in E(G)$ or $u = v$ and $ab \in E(H)$.

Note that every non-empty subset $C$ of $V(G) \times V(H)$ can be expressed as $C = \cup_{x \in S} \{(x) \times T_x\}$, where $S \subseteq V(G)$ and $T_x \subseteq V(H)$ for each $x \in S$.

**Theorem 6.** Let $G$ and $H$ be connected non-trivial graphs. A subset $C = \cup_{x \in S} \{(x) \times T_x\}$ of $V(G[H])$ is a global hop dominating set of $G[H]$ if and only if the each following conditions holds:

(i) $S$ is both a dominating and a hop dominating set of $G$.

(ii) $T_x$ is a pointwise non-dominating set of $H$ for each $x \in S$ with $|N_G(x, 2) \cap S| = 0$.

(iii) $T_x$ is a dominating set of $H$ for each $x \in S$ with $S \cap N_G(x) = \emptyset$ or $|V(G) \setminus N_G(x) \cap [V(G) \setminus N_G(y)] = \emptyset$ for each $y \in S \cap N_G(x)$. If, in addition, $N_G[x] = V(G)$, then $T_x$ is a pairwise non-dominating set of $H$.

(iv) For each $z \in V(G) \setminus S$, there exists $y \in S \cap N_G(z)$ such that $[V(G) \setminus N_G(z)] \cap [V(G) \setminus N_G(y)] \neq \emptyset$.

**Proof.** Suppose $C$ is a global hop dominating set of $G[H]$. Let $u \in V(G) \setminus S$ and pick any $a \in V(H)$. Since $C$ is a hop dominating set of $G[H]$ and $(u, a) \notin C$, there exists $(y, b) \in C$ such that $d_{G[H]}((u, a)(y, b)) = 2$. This implies that $y \in S$ and $d_C(u, y) = 2$.

Also, since $C$ is a hop dominating set of $G[H]$ and $(u, a) \notin C$, there exists $(z, c) \in C$ such that $d_{G[H]}((u, a)(z, c)) = 2$. It follows that $z \in S$ and $d_C(u, z) = 1$. Hence, $S$ is both a dominating and a hop dominating set of $G$, showing that (i) holds.

Let $x \in S$. Suppose that $|N_G(x, 2) \cap S| = 0$. Then $T_x$ is a pointwise non-dominating set of $H$. Hence, (ii) holds. Suppose now that $S \cap N_G(x) = \emptyset$ or $|V(G) \setminus N_G(x) \cap [V(G) \setminus N_G(y)] = \emptyset$ for each $y \in S \cap N_G(x)$. Let $p \in V(H) \setminus T_x$. Since $(x, p) \in V(G[H]) \setminus C$ an $C$ is a hop dominating set of $G[H]$, there exists $(w, q) \in C$ such that $d_{G[H]}((x, p)(w, q)) = 2$, that is, $d_{G[H]}((x, p)(w, q)) = 1$. If $S \cap N_G(x) = \emptyset$, then $w = x$ and $q \in T_x \cap N_H(p)$, implying that $T_x$ is a dominating set of $H$. Suppose $S \cap N_G(x) \neq \emptyset$. Suppose further that $w \neq x$. Then $w \in S \cap N_G(x)$. By assumption, $[V(G) \setminus N_G(x)] \cap [V(G) \setminus N_G(w)] = \emptyset$. Let $[(x, p), (u, t), (w, q)]$ be an $(x, p)$-$\gamma_{gh}(G[H])$ geodesic in $G[H]$. Suppose $u \neq x$. Since $xu \in E(G)$, $u \in V(G \setminus N_G(x))$. The assumption would now imply that $u \notin V(G) \setminus N_G(w)$. Thus, $u \in N_G[w]$, a contradiction. Hence, $u = x$. This, however, is not possible because $xw \in E(G)$. Therefore, $w = x$, implying that $q \in T_x \cap N_H(p)$. Hence, $T_x$ is a dominating
set of $H$. Finally, suppose that $N_G[x] = V(G)$. Then $u = x$ and $t \in N_G[p] \cap N_G[q]$. It follows that $t \notin N_H[p, q]$. Thus, $T_x$ is a pairwise non-dominating set of $H$. Therefore, (iii) holds.

Now let $z \in V(G) \setminus S$. Choose any $b \in V(H)$. Since $C$ is a hop dominating set of $G[H]$, there exists $(y, c) \in C$ such that $d_{G[H]}((z, b)(y, c)) = 2$, that is, $d_{G[H]}((z, b)(y, c)) = 1$. Hence, $y \in S \cap N_G(z)$. Let $((z, b), (s, d), (y, c))$ be a $(z, b)-(y, c)$ geodesic in $G[H]$. Then $s \in [V(G) \cap N_G(z)] \setminus [V(G) \cap N_G(y)]$, showing that (iv) holds.

For the converse, suppose that $C$ satisfies properties (i), (ii), (iii), and (iv). By (i) and (ii), $C$ is a hop dominating set of $G[H]$. Let $(v, a) \in G[H] \setminus C$ and consider the following cases:

Case 1. $v \notin S$

By (iv), let $y \in S \cap N_G(v)$ and let $u \in [V(G) \setminus N_G(v)] \setminus [V(G) \setminus N_G(y)] = \emptyset$. Let $p \in T_y$. Then $(y, p) \in C$ and $[(v, a), (u, a), (y, p)]$ is a $(v, a)-(y, p)$ geodesic in $G[H]$. Thus, $d_{G[H]}((v, a)(y, p)) = 2$.

Case 2. $v \in S$

Suppose $S \cap N_G(v) \neq \emptyset$ and $[V(G) \setminus N_G(v)] \setminus [V(G) \setminus N_G(y)] \neq \emptyset$ for some $y \in S \cap N_G(v)$. Choose any $q \in T_y$ and let $w \in [V(G) \setminus N_G(v)] \setminus [V(G) \setminus N_G(y)]$. Then $(y, q) \in C$ and $[(v, a), (w, a), (y, q)]$ is a $(v, a)-(y, q)$ geodesic in $G[H]$. Thus, $d_{G[H]}((v, a)(y, q)) = 2$. Next, suppose that $S \cap N_G(v) = \emptyset$ or $[V(G) \setminus N_G(v)] \setminus [V(G) \setminus N_G(y)] = \emptyset$ for all $y \in S \cap N_G(v)$. Suppose $N_G[v] = V(G)$. Then $T_v$ is a pairwise non-dominating set of $H$. (iii) Hence, there exists $t \in T_v \cap N_H(a)$ such that $N_H((a, d)) \neq V(H)$. This implies that $(v, d) \in C$ and there exists $t \in V(H) \setminus N_H((a, d))$. Hence, $[(v, a), (v, t), (v, d)]$ is a $(v, a)-(v, d)$ geodesic in $G[H]$, that is, $d_{G[H]}((v, a)(v, d)) = 2$. Suppose $N_G[v] \neq V(G)$. By (iii), $T_v$ is a dominating set of $H$. Again, let $d \in T_v \cap N_H(a)$ and pick $w \in V(H) \setminus N_G[v]$. Then $(v, d) \in C$ and $[(v, a), (w, a), (v, d)]$ is a $(v, a)-(v, d)$ geodesic in $G[H]$. Thus, $d_{G[H]}((v, a)(v, d)) = 2$.

Therefore, $C$ is a hop dominating set of $G[H]$. Accordingly, $C$ is a global hop dominating set of $G[H]$.

A set $S \subseteq V(G)$ is said to be dominated complement-neighborhood intersecting (dcni) (resp. total dominated complement-neighborhood intersecting (tdcni)) set of a graph $G$ if for each $v \in V(G) \setminus S$ (resp. for each $v \in S$), there exists $w \in S \cap N_G(v)$ such that $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w)) \neq \emptyset$. Let

$$
\gamma_{cn}^h(G) = \min \{|S| : S \text{ is a dcni hop dominating set of } G\},
$$

$$
\gamma_{tcn}^h(G) = \min \{|S| : S \text{ is a tdcni set of } G\}.
$$

Any dcni hop dominating set of $G$ with cardinality $\gamma_{cn}^h(G)$ is called a $\gamma_{cn}^h$-set of $G$ and any tdcni set of $G$ with cardinality $\gamma_{tcn}^h(G)$ is called a $\gamma_{tcn}$-set of $G$.

Observe that for any graph $G$, the vertex set $V(G)$ is a dominating complement-neighborhood intersecting and hop dominating set of $G$. Also, if $G_1$ is the graph obtained from the cycle $C_4 = \{a, b, c, d, a\}$ by adding the edges $av$ and $bw$, then $S = \{a, b\}$ is a dcni
Theorem 1. Let $G$ be a graph without isolated vertices.

(i) If $G$ is disconnected, then $G$ admits a $tdcni$ set.

(ii) If $G$ admits a $tdcni$ set, then $3 \leq \gamma_{tcni}(G) \leq |V(G)|$.

(iii) If $\gamma(G) \neq 2$, then $G$ admits a $tdcni$ set. If, in addition, $G$ has at most one vertex of degree one, then $\gamma_{tcni}(G) \leq |V(G)| - 1$.

Proof. (i) Suppose $G$ is disconnected and let $S = V(G)$. Let $v \in S$. Since $G$ has no isolated vertices, there exists $w \in S \cap N_G(v)$. Let $C_1$ and $C_2$ be connected components of $G$ with $w, v \in C_1$. Pick any $z \in C_2$. Then $z \in (V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w))$. Hence, $S = V(G)$ is a $tdcni$ set of $G$.

(ii) Suppose $G$ admits a $tcnid$ set. Since a $tdcni$ set is a total dominating set, it follows that $2 \leq \gamma_{tcni}(G) \leq n$. Suppose $\gamma_{tcni}(G) = 2$, say $S = \{x, y\}$ is a $\gamma_{tcni}$-set of $G$. Since $S$ is a dominating set, $V(G) \setminus S \subseteq N_G(x, y)$. Hence, $(V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(y)) = \emptyset$, contrary to the assumption that $S$ is a $tdcni$ set. Thus, $3 \geq \gamma_{tcni}(G)$.

(iii) Suppose $\gamma(G) \neq 2$. Let $v \in V(G)$ and let $w \in V(G) \cap N_G(v)$. By assumption, $N_G\{v, w\} \neq V(G)$. This implies that there exists $y \in (V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w))$, showing that $V(G)$ is a $tdcni$ set of $G$. Suppose further that $G$ has at most one vertex of degree one. Let $v \in V(G)$ such that $\delta(G) = \text{deg}_G(v)$ and let $S = V(G) \setminus \{v\}$. Note that if $\text{deg}_G(v) = 1$, then $\text{deg}_G(w) \geq 2$ for all $w \in V(G) \setminus \{v\}$. Let $u \in S \cap N_G(v)$. Since $\gamma(G) \neq 2$, $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w)) \neq \emptyset$. Let $z \in S$. Since $\text{deg}_G(z) \geq 2$, there exists $y \in S \cap N_G(z)$. Again, since $\gamma(G) \neq 2$, $(V(G) \setminus N_G(z)) \cap (V(G) \setminus N_G(y)) \neq \emptyset$. This implies that $S$ is a $tdcni$ set and $\gamma_{tcni}(G) \leq |S| = |V(G)| - 1$. □

Corollary 4. Let $G$ and $H$ be non-trivial connected graphs.

(i) If $\gamma(H) = 1$, then $\gamma_{gh}(G[H]) \leq \gamma_{h, \gamma_{ppnd}(H)}$.

(ii) If $\gamma(G) \neq 1$, then $\gamma_{gh}(G[H]) \leq \gamma_{h, \gamma_{ppnd}(H)}$.

Proof. Let $S$ be a $\gamma_{h, \gamma_{ppnd}}$-set of $G$. Let $D_1$ and $D_2$ be, respectively, a $\gamma_{ppnd}$-set and a $\gamma_{ppnd}$-set of $H$. Set $T_x = D$ for each $x \in S$ and $R_x = D_2$. If $\gamma(G) = 1$, then $C_1 = \cup_{x \in \gamma(G[H])} |S| = |S| \cap D_1$ is a global hop dominating set of $G[H]$ by Theorem 6. Hence, $\gamma_{gh}(G[H]) \leq |C_1| = |S| \cap D_1 = \gamma_{h, \gamma_{ppnd}(H)}$, proving that (i) holds. If $\gamma(G) \neq 1$, then $C_2 = \cup_{x \in S} |x \cap R_x| = S \times D_2$ is a global hop dominating set of $G[H]$ by Theorem 6. Hence, $\gamma_{gh}(G[H]) \leq |C_2| = |S| \cap D_2 = \gamma_{h, \gamma_{ppnd}(H)}$, showing that (ii) holds. □

Remark 3. The bounds in Corollary 4 are sharp.
To see this, let $G_1$ be the graph obtained from the cycle $C_4 = [a, b, c, d, a]$ by adding the edges $av$ and $bw$, and let $H = P_3$. As pointed out earlier, $S = \{a, b\}$ is a dcri hop dominating set of $G_1$. In fact, $\gamma_{\text{cri}}(G_1) = |S| = 2$. Now, $\gamma_{\text{pnd}}(H) = 2$ by Theorem 3.(iii). It can easily be verified that $\gamma_{gh}(G[H]) = 4 = \gamma_{\text{cri}}^h(P_3, \gamma_{\text{pnd}}(H))$. Also, $\gamma_{gh}(P_3[P_2]) = \gamma_{\text{cri}}^h(P_3, \gamma_{\text{pnd}}(P_2)) = 2(2) = 4$ and $\gamma_{gh}(P_2[P_2]) = \gamma_{gh}(K_4) = \gamma_{\text{cri}}^h(P_4, \gamma_{\text{pnd}}(P_2)) = 2(2) = 4$.

The Cartesian product of graphs $G$ and $H$, denoted by $G \Box H$, is the graph with vertex set $V(G \Box H) = V(G) \times V(H)$ such that $(v, p)(u, q) \in E(G \Box H)$ if and only if $uv \in E(G)$ and $p = q \in E(H)$ or $uv \in E(G)$ and $p = q \in E(H)$.  

**Theorem 7.** Let $G$ and $H$ be connected non-trivial graphs. A subset $C = \bigcup_{x \in S}\{x\} \times T_x$ of $V(G \Box H)$ is a global hop dominating set of $G \Box H$ if and only if the following conditions hold:

(i) For each $x \in V(G) \setminus S$ and for each $p \in V(H)$,

(a) there exists $y \in S \cap N_G(x)$ such that $T_y \cap N_H(p) \neq \emptyset$ or there exists $z \in S \cap N_G(x, 2)$ such that $p \in T_z$,

(b) there exists $w \in S \cap N_G(x)$ such that $p \in T_w$ and $[N_H(p) \neq V(H)$ or $(V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(w)) \neq \emptyset]$, 

(ii) For each $v \in S$ and for each $p \in V(H) \setminus T_v$, the following statements are satisfied:

(c) $N_H(p, 2) \cap T_v \neq \emptyset$ or there exists $y \in S \cap N_G(v)$ such that $T_y \cap N_H(p) \neq \emptyset$, or there exists $z \in S \cap N_G(v, 2)$ such that $p \in T_z$.

(d) $N_H(p) \cap T_v \neq \emptyset$ and $[V(G) \setminus N_G[v] \neq \emptyset$ or $|V(H)| \geq 3$] or there exists $u \in S \cap N_G(v)$ such that $p \in T_u$ and $[N_H(p) \neq V(H)$ or $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(w)) \neq \emptyset]$.

**Proof.** Suppose $C$ is a global hop dominating set of $G \Box H$. Let $x \in V(G) \setminus S$ and let $p \in V(H)$. Since $C$ is a hop dominating set of $G \Box H$ and $(x, p) \notin C$, there exists $(y, q) \in C$ such that $d_{G \Box H}((x, p)(y, q)) = 2$. Since $y \in S$, $x \neq y$. If $xy \notin E(G)$, then $pq \notin E(H)$. Hence, $q \in T_y \cap N_H(p)$. So suppose that $y \notin N_G(x)$. Since $d_{G \Box H}((x, p)(y, q)) = 2$, it follows that $y \in N_G(x, 2)$ and $p \in T_y$. Hence, $p \notin T_y$, showing that (a) holds. Now, since $C$ is also a hop dominating set of $G \Box H$, there exists $(w, t) \in C$ such that $d_{G \Box H}((x, p)(w, t)) = 2$. It follows that $d_{G \Box H}((x, p)(w, t)) = 1$. This implies that $w \in S \cap N_G(x)$ and $p \in T_w$. Now, if $[(x, p), (z, s), (w, t)]$ is an $(x, p)$-$s(w, t)$ geodesic in $G \Box H$, then $s \in V(H) \setminus N_H[p]$ or $z \in ((V(G) \setminus N_G(x)) \cap (V(G) \setminus N_G(w))$. This shows that (b) holds.

Next, let $v \in S$ and let $p \in V(H) \setminus T_v$. Since $C$ is a hop dominating set of $G \Box H$ and $(v, p) \notin C$, there exists $(y, q) \in C$ such that $d_{G \Box H}((v, p)(y, q)) = 2$. Suppose $y = v$. Then $d_H(p, q) = 2$ and so $q \in N_G(p, 2) \cap T_v$. Suppose $y \neq v$. If $d_G(y, v) = 1$, then $y \in S \cap N_G(v)$ and $d_H(p, q) = 1$, i.e. $q \in T_y \cap N_H(p)$. If $d_G(y, v) \neq 1$, then $d_G(y, v) = 2$. Hence, $y \in S \cap N_G(v, 2)$ and $p = q$, that is, $p \in T_y$. Thus, (b) holds.

On the other hand, since $C$ is also a hop dominating set of $G \Box H$ and $(v, p) \notin V(G \Box H) \setminus C$, there exists $(u, t) \in C$ such that $d_{G \Box H}((v, p)(u, t)) = 2$. Again, this would imply
that $d_{G \square H}((v,p)(u,t)) = 1$. If $u = v$, then $t \in N_H(p) \cap T_v$. Since $d_{G \square H}((v,p)(u,t)) = d_{G \square H}((v,p)(v,t)) = 2$, $V(G) \setminus N_G[v] \neq \emptyset$ or $|V(H)| \geq 3$. Suppose $u \neq v$. Then $u \in S \cap N_G(v) \cap T_v$. Since $d_{G \square H}((v,p)(u,t)) = 2$, $V(H) \setminus N_H(p) \neq \emptyset$ or $(V(G) \setminus N_G(v)) \cap (V(G) \setminus N_G(u)) \neq \emptyset$.

For the converse, suppose that $C$ satisfies properties (i) and (ii). Let $(v,p) \in V(G[H]) \setminus C$ and consider the following cases:

Case 1. $v \notin S$

By the assumption that (a) of (i) holds, suppose first that there exists $y \in S \cap N_G(x)$ such that $T_y \cap N_H(p) \neq \emptyset$. Let $q \in T_y \cap N_H(p) \neq \emptyset$. Then $(y,q) \in C$ and $d_{G \square H}((v,p)(y,q)) = d_G(v,y) + d_H(p,q) = 2$. Next, suppose that there exists $z \in S \cap N_G(v)$ such that $p \in T_z$. Then $(z,p) \in C$ and $d_{G \square H}((v,p)(z,p)) = d_G(v,z) = 2$.

Since (b) of (i) also holds, suppose that there exists $w \in S \cap N_G(v)$ such that $p \in T_w$. Then $(w,p) \in C \cap N_G((v,p))$. If $N_H[p] \neq V(H)$, we may pick any $s \in V(H) \setminus N_H[p]$. Then $(w,s) \notin N_{G[H]}((v,p)) \cup N_{G[H]}((w,p))$. It follows that $[(v,p),(w,s),(w,p)]$ is a $(v,p)$-geodesic in $G \bigcirc H$. Thus, $d_{G \square H}((v,p)(w,p)) = 2$. Instead of $N_H(p) \neq V(H)$, suppose that $(V(G) \setminus N_G(x))(v,p),(v,a)$ is a $(v,a)$-geodesic in $G \bigcirc H$. Hence, $d_{G \square H}((v,p)(v+a)) = 2$. Now, using (d) of (ii), assume that $N_H(p) \cap T_v \neq \emptyset$, say $a \in N_H(p) \cap T_v$. Then $(v,a) \in C$. If there exists $w \in V(G) \setminus N_G[v]$, then $[(v,p),(v+a),(v,a)]$ is a $(v,a)$-geodesic in $G \bigcirc H$. Thus, $d_{G \square H}((v,p)(v,a)) = 2$. If $|V(H)| \geq 3$, then we may pick any $b \in V(H) \setminus \{a,p\}$. Let $z \in N_G(v)$. Then $[(v,p),(z,b),(v,a)]$ is a $(v,a)$-geodesic in $G \bigcirc H$. Hence, $d_{G \square H}((v,p)(v+a)) = 2$. Next, assume that there exists $u \in S \cap N_G(v)$ such that $p \in T_u$. Then $(u,p) \in C$. If $V(H) \setminus N_H(p)$, then $[(v,p),(u,l),(u,p)]$ is a $(v,p)$-geodesic in $G \bigcirc H$. This implies that $d_{G \square H}((v,p)(u,p)) = 2$. If there exists $z \in (V(G) \setminus N_G(v))(v,p),(v,u)$, then $[(v,p),(z,p),(u,p)]$ is a $(v,p)$-geodesic in $G \bigcirc H$, implying that $d_{G \square H}((v,p)(u,p)) = 2$.

Therefore, $C$ is a hop dominating set of $G \bigcirc H$. Accordingly, $C$ is a global hop dominating set of $G \bigcirc H$.

**Corollary 5.** Let $G$ and $H$ be non-trivial connected graphs.

(i) If $\gamma(H) = 1$, then $\gamma_{gh}(G \bigcirc H) \leq |V(H)| \cdot \gamma_{tcni}(G)$.

(ii) If $\gamma(H) \neq 1$, then $\gamma_{gh}(G \bigcirc H) \leq |V(H)| \cdot \gamma_{tl}(G)$.

**Proof.** Let $S$ be a $\gamma_{tcni}$-set of $G$ and let $T_x = V(H)$ for all $x \in S$. Let $C = \bigcup_{x \in S} \{x\} \times T_x = S \times V(H)$. If $\gamma(H) = 1$, then $C$ is a global hop dominating set of $G \bigcirc H$ by Theorem 7. Thus, $\gamma_{gh}(G \bigcirc H) \leq |C| = |V(H)| \cdot \gamma_{tcni}(G)$. 


Next, let $S'$ be a $\gamma_t$-set of $G$ and let $R_x = V(H)$ for all $x \in S'$. Let $C' = \bigcup_{x \in S'} \{x\} \times R_x = S \times V(H)$. If $\gamma(H) \neq 1$, then $C'$ is a global hop dominating set of $G \square H$ by Theorem 7. This implies that $\gamma_{gh}(G \square H) \leq |C'| = |V(H)| \gamma_t(G)$.

3. Conclusion

The global hop dominating sets in the join, corona, lexicographic product, and the Cartesian product of two graphs have been characterized. From these characterizations, we determined either the exact values or upper bounds of the global hop domination numbers of the corresponding graphs.

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