On Tosha-degree of an edge in a graph

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Abstract. In an earlier paper, we have introduced the Tosha-degree of an edge in a graph without multiple edges and studied some properties. In this paper, we extend the definition of Tosha-degree of an edge in a graph in which multiple edges are allowed. Also, we introduce the concepts - zero edges in a graph, T-line graph of a multigraph, Tosha-adjacency matrix, Tosha-energy, edge-adjacency matrix and edge energy of a graph G and obtain some results.

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1. Introduction

For standard terminology and notion in graphs and matrices, we refer the reader to the text-books of Harary [2] and Bapat [1]. The non-standard will be given in this paper as and when required.

Throughout this paper, \( G = (V, E) \) denotes a graph (finite and undirected) and \( V = V(G) \) and \( E = E(G) \) denote vertex set and edge set of \( G \), respectively. The degree of a vertex \( v \in V(G) \), denoted by \( d(v) \) or \( d_G(v) \), is the number of edges incident on \( v \), with self-loops counted twice. A vertex of degree one is a pendant vertex and an edge incident onto a pendant vertex is a pendant edge. A graph \( G \) is \( r \)-regular if every vertex...
of $G$ has degree $r$. The minimum degree $\delta(G)$ of a graph $G$ is the minimum degree among all the vertices of $G$ and the maximum degree $\Delta(G)$ of $G$ is the maximum degree among all the vertices of $G$.

Two non-distinct edges in a graph are adjacent if they are incident on a common vertex. We consider that an edge in a graph is not adjacent to itself. The letters $k, l, m, n,$ and $r$ denote positive integers or zero.

The line graph $L(G)$ of a simple graph with at least one edge is the graph $(W, F)$, where there is a one-to-one correspondence $\phi$ from $E$ to $W$ such that there is an edge between $\phi(\alpha)$ and $\phi(\beta)$ if and only if the edges $\alpha$ and $\beta$ are adjacent. We identify the set $W$ by $E$.

The adjacency matrix of a graph $G$ with $n$ vertices is denoted by $A(G)$. If $A(G)$ is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A(G)$, the energy of $G$ is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$
2. Tosha-degree of an edge in a graph

In [4], R. Rajendra and P. S. K. Reddy have defined the Tosha-degree of an edge in a graph without multiple edges as follows: The Tosha-degree of an edge $\alpha$ in a graph $G$ without multiple edges, denoted by $T(\alpha)$, is the number of edges adjacent to $\alpha$ in $G$, with self-loops counted twice. Here we allow graphs with multiple edges (multi-graphs) and the new definition of the Tosha-degree of an edge in a graph (with or without multiple edges) is given below:

Definition 1. Let $\alpha$ be an edge in a graph $G$. The Tosha-degree of $\alpha$, denoted by $T(\alpha)$ or $T_G(\alpha)$, is the number of edges adjacent to $\alpha$ in $G$, where self-loops and edges parallel to $\alpha$ are counted twice.

By the Definition 1, for any edge $\alpha$ in a graph $G$, $T(\alpha) \geq 0$.

Definition 2. A graph $G$ is said to be a Tosha-regular graph if all edges are of equal Tosha-degree. We say that $G$ is $l$-Tosha-regular, if $T(\alpha) = l$, for all $\alpha \in E(G)$.

The following proposition is proved for graphs without parallel edges in [4]. This result is true for graphs having parallel edges also with respect to the Definition 1.

Proposition 1. [4] Let $\alpha$ be an edge in a graph $G$ with end vertices $u$ and $v$.

(i) If $\alpha$ is not a self-loop, then

$$T(\alpha) = d(u) + d(v) - 2$$

(ii) If $\alpha$ is a self-loop, then $u = v$ and

$$T(\alpha) = d(u) - 2$$

Proof. The proof follows by the definition 1, and the definition of degree of a vertex.

Observation: By the Proposition 1, for an edge $\alpha$ in a graph $G$, it follows that,

(a) if $\alpha$ is not a self-loop, then

$$2(\delta(G) - 1) \leq T(\alpha) \leq 2(\Delta(G) - 1);$$

(b) if $\alpha$ is a self-loop, then

$$\delta(G) - 2 \leq T(\alpha) \leq \Delta(G) - 2.$$ 

Corollary 1. [4] If $G$ is a simple graph and $\alpha$ is an edge in $G$, then

$$T(\alpha) = d_{L(G)}(\alpha)$$

where $d_{L(G)}(\alpha)$ is the degree of $\alpha$ as a vertex in the line graph $L(G)$ of $G$.

Proof. Follows from the definition of $L(G)$ and Eq.(1).
Corollary 2. In a simple graph $G$, the number of odd Tosha-degree edges is even.

Proof. In any graph the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph $L(G)$ of $G$ is even. Since the vertices in $L(G)$ are corresponding to the edges in $G$, by Eq.(3) it follows that, the number of odd Tosha-degree edges in $G$ is even.

Remark 1. The Corollary 2 may not be true for the graphs having self-loops. There are graphs with odd number of edges and all edges are of odd Tosha-degree. For eg., consider the graph $G$ given in Figure 1. The graph $G$ has three edges namely, $\alpha$, $\beta$ and $\gamma$. We observe that $T(\alpha) = 1$, $T(\beta) = 3$, $T(\gamma) = 1$ and hence all the edges in $G$ are of odd Tosha-degree.

![Figure 1: Graph containing odd number of odd Tosha-degree edges.](image)

Observation: Let $\alpha$ be an edge in a simple graph $G$. The addition of a parallel edge $\beta$ to $\alpha$ gives a count plus two to the Tosha-degree of $\alpha$ and to the edges parallel to $\alpha$, and a count plus one to non-parallel edges adjacent to $\alpha$ in the new graph $G + \beta$ and Tosha-degrees of all other edges are unaltered $G + \beta$. Hence an odd (even) Thosha-degree edge $\gamma$ remains odd (even) Tosha-degree in $G + \beta$, if it is not adjacent to $\alpha$ or $\gamma = \alpha$ in $G$.

Corollary 3. If $\alpha$ and $\beta$ are parallel edges in a graph $G$, then $T(\alpha) = T(\beta)$ in $G$.

Proof. The proof follows by Proposition 1.

2.1. $T$-line graph of a multigraph

Definition 3. A multigraph is a graph in which multiple edges (parallel edges) are permitted between any pair of vertices. All multigraphs in this paper are loopless.

We say that two distinct edges $\alpha$ and $\beta$ in a multigraph $G$ are $k$-adjacent if they are adjacent and share $k$ end vertices.

We say that two distinct vertices $u$ and $v$ in a multigraph $G$ are $r$-adjacent if they are adjacent and the number of edges between them is $r$ (i.e., $r$ edges have common end vertices $u$ and $v$).

From the Definition 3, it follows that, when two distinct edges $\alpha$ and $\beta$ are $k$-adjacent in a multigraph $G$, we have,

$$k = \begin{cases} 1, & \text{if } \alpha \text{ and } \beta \text{ are not parallel;} \\ 2, & \text{if } \alpha \text{ and } \beta \text{ are parallel.} \end{cases}$$
Definition 4. Given a multigraph \( G = (V, E) \), the T-line graph of \( G \) denoted by \( TL(G) \), is a graph with vertex set \( E \); two distinct vertices \( \alpha \) and \( \beta \) are \( k \)-adjacent in \( TL(G) \) if and only if their corresponding edges in \( G \) are \( k \)-adjacent.

From the Definition 4, it is clear that,

(a) \( TL(G) \) is also a multigraph,

(b) if \( G \) is a simple graph, then \( TL(G) \) is nothing but \( L(G) \).

Proposition 2. Let \( G \) be a multigraph and \( \alpha \) be a vertex in \( TL(G) \) (so \( \alpha \) is an edge in \( G \)). Then

\[
d_{TL(G)}(\alpha) = d_G(u) + d_G(v) - 2 = T_G(\alpha)
\]

where \( u \) and \( v \) are end vertices of \( \alpha \) in \( G \).

Proof. Proof follows by the definitions 1, 3 and 4, and propositions 1 and 2.

Corollary 4. In a multigraph \( G \), the number of odd Tosha-degree edges is even.

Proof. In any graph(multigraph) the number of odd degree vertices is even. So, the number of odd degree vertices in the line graph \( TL(G) \) of \( G \) is even. Since the vertices in \( TL(G) \) are corresponding to the edges in \( G \), by Eq.(4), the number of odd Tosha-degree edges in \( G \) is even.

3. Zero edges in a graph

Definition 5. In a graph \( G \), an edge \( \alpha \) is said to be a zero edge if its Tosha degree is zero i.e., \( T(\alpha) = 0 \).

Observations: The edge in the complete graph \( K_2 \) is a zero edge. The self-loop in the graph containing only one vertex and a self-loop attached to that vertex, is a zero edge.

Proposition 3. A simple connected graph \( G \) has a zero edge if and only if \( G \cong K_2 \).

Proof. Suppose that \( G \) is a simple connected graph having a zero edge, say \( \alpha = uv \), where \( u \) and \( v \) are end vertices of \( \alpha \). Then

\[
d(u) + d(v) - 2 = 0
\]

Since \( G \) is connected, \( d(u) \geq 1 \) and \( d(v) \geq 1 \); from Eq.(5), \( d(u) = 1 \) and \( d(v) = 1 \). Therefore, there is no other edge in \( G \) incident to \( u \) and \( v \). So \( G \) has only one edge \( \alpha \). Since \( G \) is connected, \( G \cong K_2 \).

Conversely, if \( G \cong K_2 \), then clearly \( G \) is a simple connected graph having only one edge whose Tosha-degree is zero.
Corollary 5. A simple connected graph $G$ with two or more edges, has no zero edge. Hence $T(\alpha) \geq 1$, $\forall \alpha \in E(G)$.

Proof. Follows from Proposition 3.

Corollary 6. A simple graph $G$ has no zero edge if and only if either $G \not\cong K_2$ or no component of $G$ is isomorphic to $K_2$ or no component of $G$ is of only one vertex with a self-loop.

Proof. Follows from Proposition 3.

4. Degree colorable graphs

In this section we consider self-loop free graphs (multigraphs).

Definition 6. A graph $G$ is degree colorable if no two adjacent vertices have the same degree.

Theorem 1. If all the edges of a graph $G$ are of odd Tosha-degree, then $G$ is a degree colorable graph with even number of vertices.

Proof. Suppose that $G$ is a graph in which all the edges are of odd Tosha-degree. By the corollaries 2 and 4, it follows that $G$ has an even number of vertices. Let $\alpha$ be an edge in $G$ with end vertices $u$ and $v$. Then by Eq.(1) and Eq.(4),

$$T(\alpha) = d(u) + d(v) - 2.$$ 

Since $T(\alpha)$ is odd, $d(u) \neq d(v)$. Thus, no two adjacent vertices in $G$ have the same degree. Therefore $G$ is a degree colorable graph.

By Theorem 1, the following corollary is immediate.

Corollary 7. An $l$-Tosha-regular graph, where $l$ is an odd positive integer, is degree colorable.

Remark 2. There are degree colorable non-Tosha-regular graphs with odd number of vertices. The following graph is an example for such graphs, in which the edges are indicated by respective Tosha-degrees.

5. Tosha-even graphs

Definition 7. A graph $G$ is said to be Tosha-even if all its edges are of even Tosha-degree.

We recall the following proposition from [4].

Proposition 4. [4, Proposition 2.15] If $G$ is an Euler graph, then all edges in $G$ are of even Tosha-degree.
Corollary 8. Euler graphs are Tosha-even.

Proof. Follows from the Proposition 4.

Remark 3. The converse of the Corollary 8 is not true in general. There are connected graphs with even number of vertices and all vertices are of odd degree, for instance, $K_4$. Such graphs are not Euler graphs, but are Tosha-even.

Proposition 5. There exist degree colorable Tosha-even graphs that are not Euler graphs.

Proof. The following graph $G$ (see Figure 3) is an example of a degree colorable Tosha-even graph which is not an Euler graph. In $G$, the vertices and edges are indicated by their degrees and Tosha-degrees, respectively. We see that all vertices of $G$ are of odd degree and hence $G$ is not an Euler graph. But all edges are of Tosha-even, so $G$ is a Tosha-even graph.

6. Tosha-adjacency matrix of a graph

Definition 8. If $G$ is a graph with $n$ vertices $v_1, \ldots, v_n$ and no parallel edges. The Tosha-adjacency matrix of the graph $G$ is an $n \times n$ matrix $A_T(G) = (t_{ij})$ defined over the ring of integers such that

$$t_{ij} = \begin{cases} T(v_iv_j), & \text{if } v_iv_j \in E \\ 0, & \text{otherwise.} \end{cases}$$

Observations:

![Figure 2: A degree colorable non-Tosha-regular graph with 3 vertices.](image)

![Figure 3: A degree colorable Tosha-even graph which is not an Euler graph.](image)
(i) By the definition of the Tosha-degree of an edge, we have

\[ T(v_i v_j) = \begin{cases} 
  d(v_i) + d(v_j) - 2, & \text{if } v_i v_j \in E \text{ and } i \neq j; \\
  d(v_i) - 2, & \text{if } v_i v_j \in E \text{ and } i = j; \\
  0, & \text{if } v_i v_j \notin E.
\end{cases} \]

Therefore, \( t_{ij} = t_{ji} \). Therefore \( A_T(G) \) is a real symmetric matrix.

(ii) The entries along the principal diagonal of \( A_T(G) \) are all 0s if and only if either \( G \) has no self-loops or \( G \) has only self loops that are zero edges. Hence if either \( G \) has no self-loops or \( G \) has only self loops that are zero edges, then \( tr(A_T(G)) = 0 \). In this case, if \( \mu_1, \mu_2, \ldots, \mu_n \) are the eigenvalues of \( A_T(G) \), then

\[ \sum_{i=1}^{n} \mu_i = 0. \]

(iii) If \( G \) has no zero edges, then the degree of a vertex equals the number of non-zero entries in the corresponding row or column; and the non-zero entry in the \( ij \)-th place gives the Tosha-degree of the corresponding edge incident to \( i \)-th and \( j \)-th vertices.

(iv) For a zero edge free graph \( G \), the adjacency matrix \( A(G) \) can be obtained from the Tosha-adjacency matrix \( A_T(G) \) by replacing all the non-zero entries by 1s. This is possible because, in a zero edge free graph Tosha-degrees of edges are non-zero. Thus, reconstruction of the graph from the Tosha-adjacency matrix is possible if the given graph has no zero edges.

Throughout this section \( G \) denotes a graph with no parallel edges.

**Theorem 2.** If a graph \( G \) with \( n \) vertices is \( l \)-Tosha-regular, then

\[ A_T(G) = l \cdot A(G). \]

**Proof.** Suppose that \( G \) is \( l \)-Tosha-regular. Then \( T(\alpha) = l \), for all \( \alpha \in E(G) \). Let \( A(G) = (a_{ij}) \) and \( A_T(G) = (t_{ij}) \) be the adjacency matrix and the Tosha-adjacency matrix of \( G \), respectively. Then by the definition of the Tosha-adjacency matrix \( A_T(G) \), we have

\[ t_{ij} = \begin{cases} 
  l, & \text{if } v_i v_j \in E \\
  0, & \text{otherwise}
\end{cases} = l \cdot a_{ij}. \]

Therefore, \( A_T(G) = l \cdot A(G). \)
Corollary 9. If a graph $G$ with $n$ vertices is $r$-regular, then

$$A_T(G) = 2(r - 1)A(G).$$

Proof. If a graph $G$ with $n$ vertices is $r$-regular, then $G$ is $2(r - 1)$-Tosha-regular (by [4, Corollary 2.6]) and hence by Theorem 2, $A_T(G) = 2(r - 1)A(G)$.

Corollary 10. A graph $G$ is 1-Tosha-regular if and only if $A_T(G) = A(G)$.

Proof. ($\Leftarrow$:) Suppose that for a graph $G$, $A_T(G) = A(G)$. Then by the definitions of $A_T(G)$ and $A(G)$, it follows that, $T(\alpha) = 1$, $\forall \alpha \in E(G)$. Hence, $G$ is 1-Tosha-regular.

($\Rightarrow$:) Follows by Theorem 2.

7. Tosha-energy of a graph

Definition 9. Let $G$ be graph with $n$ vertices $v_1, \ldots, v_n$ and no parallel edges. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the Tosha-adjacency matrix $A_T(G)$ of $G$. The Tosha-energy of $G$, denoted by $\mathcal{E}_T(G)$, is defined as

$$\mathcal{E}_T(G) = \sum_{i=1}^{n} |\mu_i|. \quad (6)$$

Throughout this section $G$ denotes a graph with no parallel edges.

Proposition 6. The Tosha-energy of an $l$-Tosha-regular graph $G$ with $n$ vertices is given by

$$\mathcal{E}_T(G) = l \cdot \mathcal{E}(G) \quad (7)$$

where $\mathcal{E}(G)$ is the energy of $G$.

Proof. Let $G$ be an $l$-Tosha-regular graph with $n$ vertices. Then by the Theorem 2, the Tosha-adjacency matrix of $G$ is

$$A_T(G) = l \cdot A(G) \quad (8)$$

where $A(G)$ is the adjacency matrix of $G$. For brevity we write $A$ for $A(G)$ and $A_T$ for $A_T(G)$. We consider two cases: (i) When $l > 0$ and (i) When $l = 0$.

Case (i): When $l > 0$. Let $\mu$ be an eigenvalue of $A_T$. From Eq.(8) we have,

$$\det(A_T - \mu I) = 0 \iff \det(lA - \mu I) = 0$$

$$\iff l^n \det \left( A - \frac{\mu}{l}I \right) = 0$$

$$\iff \det \left( A - \frac{\mu}{l}I \right) = 0.$$
Therefore, $\mu$ is an eigenvalue of $A_T$ if and only if $\frac{\mu}{l}$ is an eigenvalue of $A$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of the $A_T$. Then $\frac{\mu_1}{l}, \frac{\mu_2}{l}, \ldots, \frac{\mu_n}{l}$ are the eigenvalues of $A$ and the Tosha-energy of $G$ is

$$E_T(G) = \sum_{i=1}^{n} |\mu_i|$$

$$= l \cdot \sum_{i=1}^{n} \left| \frac{\mu_i}{l} \right|$$

$$= l \cdot E(G).$$

**Case (ii): When $l = 0$.** From Eq.(7), $A_T = 0$ and so zero is the only eigenvalue of $A_T$ of multiplicity $n$. In this case, $E_T(G) = 0 = 0 \cdot E(G)$.

**Corollary 11.** The Tosha-energy of an $r$-regular graph $G$ with $n$ vertices is given by

$$E_T(G) = 2(r-1)E(G)$$

(9)

where $E(G)$ is the energy of $G$.

**Proof.** Let $G$ be an $r$-regular graph with $n$ vertices. By [4, Corollary 2.6] $G$ is a $2(r-1)$-Tosha-regular graph. Then by Proposition 6, the proof follows.

**Corollary 12.**

(i) For the complete graph $K_n$ on $n > 1$ vertices,

$$E_T(K_n) = 2(n-2)E(K_n) = 4(n-1)(n-2).$$

(ii) For the cycle graph $C_n$ on $n > 1$ vertices,

$$E_T(C_n) = 2E(C_n) = 4 \sum_{i=0}^{n-1} \cos \left( \frac{2\pi i}{n} \right)$$

(iii) For the complete bipartite graph $K_{m,n}$,

$$E_T(K_{m,n}) = (m+n-2)E(K_{m,n}) = 2(m+n-2)\sqrt{mn}.$$
Since $K_n$ is an $(n - 1)$-regular graph, from Eq.(9) we have,
\[ E_T(K_n) = 2(n - 2)E(K_n) = 2(n - 2) \cdot 2(n - 1) = 4(n - 1)(n - 2). \]

(ii) The eigen values of $A(C_n)$ are
\[ 2 \cos \left( \frac{2\pi i}{n} \right), \quad i = 0, 1, \ldots, n - 1. \]
Therefore
\[ E(C_n) = 2 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right|. \]
Since $C_n$ is an 2-regular graph, from Eq.(9) we have,
\[ E_T(C_n) = 2 \cdot E(C_n) = 4 \sum_{i=0}^{n-1} \left| \cos \left( \frac{2\pi i}{n} \right) \right|. \]

(iii) The eigen values of $A(K_n)$ are given below:

\[
\begin{array}{ccc}
\text{eigen value} & \rightarrow & \begin{pmatrix} -\sqrt{mn} & 0 & \sqrt{mn} \\ 1 & n + m - 2 & 1 \end{pmatrix} \\
\text{multiplicity} & \rightarrow & \end{array}
\]
Therefore
\[ E(K_{m,n}) = 2\sqrt{mn}. \]
Since $K_{m,n}$ is an $(m + n - 2)$-Tosha-regular graph, from Eq.(8) we have,
\[ E_T(K_{m,n}) = (m + n - 2)E(K_{m,n}) = 2(m + n - 2)\sqrt{mn}. \]

**Corollary 13.** (i) For the path $P_2$ of 2 vertices, $E_T(P_2) = 0$.

(ii) For the path $P_3$ of 3 vertices, $E_T(P_3) = E(P_3) = 2\sqrt{2}$.

**Proof.** Since $P_2 = K_{1,1}$ and $P_3 = K_{2,1}$, (i) and (ii) follow immediately from Corollary 12 (iii).

**Theorem 3.** Let $G$ be a simple connected graph with at least one edge. Then
\[ A_T(G) = A(G) \iff G = P_3. \]

**Proof.** ($\Leftarrow$) If $G = P_3$, then it has two edges and each of these are of Tosha-degree 1. Therefore, it is 1-Tosha-regular and hence by Theorem 2, $A_T(G) = A(G)$.

($\Rightarrow$) Suppose that $A_T(G) = A(G)$. Then $G$ is 1-Tosha-regular and hence
\[ T(v_iv_j) = 1, \quad \forall \; v_iv_j \in E(G) \]
\[ \Rightarrow d(v_i) + d(v_j) - 2 = 1, \; \forall \; v_iv_j \in E(G) \]
\[ \Rightarrow d(v_i) = 3 - d(v_j), \; \forall \; v_iv_j \in E(G) \]

Therefore, for any edge \( \alpha \) in \( G \) with end vertices \( u \) and \( v \),
\[ d(u) = 3 - d(v) \quad (10) \]

Since \( G \) is connected, \( d(v) > 0 \) and \( d(u) > 0 \), and from Eq.(10) we have, \( d(u) < 3 \); which implies
\[ d(u) = 1 \; \text{or} \; 2. \quad (11) \]

Let \( u \) be an arbitrary vertex in \( G \). Since \( G \) is a simple connected graph with at least one edge, \( u \) is an end vertex of at least one edge say \( \alpha \). Let \( v \) be the other end vertex of \( \alpha \) in \( G \). Then by Eq.(10) and Eq.(11), either \( d(u) = 1 \) and \( d(v) = 2 \) or \( d(u) = 1 \) and \( d(v) = 2 \).

If \( d(u) = 1 \) and \( d(v) = 2 \), there is another vertex \( w \) adjacent to \( v \) and \( d(w) = 1 \) (by above argument). There are no other vertices adjacent to the vertices \( u, v \) and \( w \). So, \( G \) is a path with 3 vertices. A similar argument can be used for the case \( d(u) = 1 \) and \( d(v) = 2 \), to show that \( G \) is \( P_3 \).

8. Edge-adjacency matrix and edge-energy of a graph

**Definition 10.** We say that two distinct edges \( \alpha \) and \( \beta \) in a graph \( G \) (where self-loops and parallel edges are allowed) are \( k \)-adjacent if they are adjacent and share \( k \) end vertices. We consider that an edge in a graph is not adjacent to itself.

**Definition 11.** If \( G \) is a graph with \( m \) edges \( e_1, \ldots, e_m \). The edge-adjacency matrix of the graph \( G \) is an \( m \times m \) matrix \( A_E(G) = (x_{ij}) \) defined over the ring of integers such that

\[ x_{ij} = \begin{cases} 
  k, & \text{if } e_i \text{ and } e_j \text{ are } k-\text{adjacent}; \\
  0, & \text{otherwise.} 
\end{cases} \]

**Observations:**

(i) \( A_E(G) \) is a \( \{0, 1, 2\} \)-matrix and it is real symmetric. If \( G \) is a simple graph, then \( A_E(G) \) is a \( \{0, 1\} \)-matrix.

(ii) The entries along the principal diagonal of \( A_E(G) \) are all 0s. Therefore, \( tr(A_E(G)) = 0 \). Hence if \( \nu_1, \nu_2, \ldots, \nu_m \) are the eigenvalues of \( A_E(G) \), then

\[ \sum_{i=1}^{m} \nu_i = 0. \]

(iii) If \( G \) has no self-loops, then the Tosha-degree of an edge equals the sum of entries in the corresponding row or column of \( A_E(G) \).
Proposition 7. For a multigraph $G$, the edge-adjacency matrix of $G$ is the adjacency matrix of the $T$-line graph of $G$. That is,

$$A_E(G) = A(TL(G)).$$

Proof. Follows by the definitions 4 and 11.

Corollary 14. For a simple graph $G$, the edge-adjacency matrix of $G$ is the adjacency matrix of the line graph of $G$. That is,

$$A_E(G) = A(L(G)).$$

Proof. For simple graph $G$, $TL(G) = L(G)$ and so by Proposition 7 the result follows.

Definition 12. Let $G$ be graph with $m$ edges $e_1, \ldots, e_m$. Let $\nu_1, \nu_2, \ldots, \nu_m$ be the eigenvalues of the edge-adjacency matrix $A_E(G)$ of $G$. The edge-energy of $G$, denoted by $E_E(G)$, is defined as

$$E_E(G) = \sum_{i=1}^{m} |\nu_i|.$$  \hfill (12)

Corollary 15. For a multigraph $G$, the edge-energy of $G$ is the energy of the $T$-line graph of $G$. That is,

$$E_E(G) = E(TL(G)).$$

Proof. Follows by Proposition 7.

Corollary 16. For a simple graph $G$, the edge-energy of $G$ is the energy of the line graph of $G$. That is,

$$E_E(G) = E(L(G)).$$

Proof. Follows by Corollary 14.

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