On Generalized $\beta$-Open Sets in Ideal Bitopological Space

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Abstract. In this article, we introduce and study the concepts of $\gamma_{ij}$-semi-$I$-open sets and $\gamma_{ij}$-$\beta I$-open sets by generalizing $(i,j)$-semi-$I$-open sets and $(ij)$-$\beta I$-open sets, respectively, in ideal bitopological spaces with an operation $\gamma : \tau \to P(X)$. Further, we describe and study $(\gamma, \delta)_{ij}$-semi-$I$-continuous and $(\gamma, \delta)_{ij}$-$\beta I$-continuous functions in ideal bitopological spaces and their related notions. In addition, various examples and counterexamples are given for answers to some questions raised in this study.

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1. Introduction

Kelly [11] in 1963, introduced the triple $(X, \tau_1, \tau_2)$ as bitopological space, where $X$ is a nonempty set, $\tau_1$ and $\tau_2$ are topologies on $X$. Levine [17] in 1963, introduced the notion of semi-open sets in bitopological spaces. Khedr [14] in 1992, defined semi-preopen ($\beta$-open) sets in bitopological spaces. K. Kuraowski [15] in 1966, studied and applied the concept of ideals on topological spaces. An ideal $I$ on a topological space $(X, \tau)$ is a collection of subsets of $X$ having the heredity property (i) if $A \in I$ and $B \subseteq A$ then $B \in I$ and (ii) if $A \in I$ and $B \in I$ then $A \cup B \in I$. Ekici [5] in 2012, studied the concept of semi-I-open sets in ideal topological spaces. If $I$ is an ideal on $X$ then $(X, \tau_1, \tau_2, I)$ is called an ideal bitopological space. Kasahara.S [10] in 1979 described an operation $\gamma$ on $\tau$ as a mapping $\gamma : \tau \to P(X)$ such that $U \subseteq U^\gamma$, for each $U \in \tau$. Khedr [12] in 1984, extended the operation $\gamma$ to bitopological space as a mapping $\gamma : \tau_1 \cup \tau_2 \to P(X)$ such that for each $U \in \tau_1 \cup \tau_2$, where $U^\gamma$ denotes the value of $\gamma$ at $U$. For example the operations $U^\gamma = U, U^\gamma = Cl_i(U), U^\gamma = Int_j(Cl_i(U))$ for $U \in \tau_j$ are operations on $\tau_1 \cup \tau_2$. Caldas [3] in 2013, introduced the notion of $\beta$-open sets in ideal bitopological spaces. Csaszar [4] in 1997, defined generalized open sets in generalized topological spaces.

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Throughout the paper, \((X, \tau_1, \tau_2)\) always mean bitopological space on with no separation axioms are supposed in this space, also \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space. Let \(A\) be a subset of \(X\), by \(\text{Int}_\gamma(A)\) [10] and \(\text{Cl}_\gamma(A)\) [20] we denote respectively the interior and closure of \(A\) with regard to \(\tau_i\) for \(i = 1, 2\).

A subset \(A\) of a bitopological space will be called a \(\gamma_i\)-open set if for each \(x \in A\), there exists an \(\tau_i\)-open set \(U\) such that \(x \in U\) and \(U^\gamma \subseteq A\). Let \(\tau_{\gamma_i}\) denotes the set of all \(\gamma_i\)-open set in \(X\). Obviously, we have \(\tau_{\gamma_i} \subseteq \tau_i\) [13]. Complement of all \(\gamma_i\)-open sets are called \(\gamma_i\)-closed. Assumed \((X, \tau_1, \tau_2, \mathcal{I})\) as an ideal bitopological space and if \(P(X)\) is the set of all subsets of \(X\), a set operator \((\cdot)^*_i : P(X) \to P(X)\) named the local function of \(A\) [22] with regard to \(\tau_i\) and \(\tau\). The definition of local function is given as: for \(A \subseteq X\), \(A^*_i(\tau_i, \mathcal{I}) = \{x \in X|U \cap A \notin \mathcal{I}, \text{for all}\ U \in \tau_i(x)\}\) where, \(\tau_i(x) = \{U \in \tau_i|x \in U\}\). Observe additionally that closure operator for \(\tau^*_i(\mathcal{I})\) accurate than \(\tau_i\) is defined by \(\text{Cl}^*_i(A) = A \cup A^*_i\).
The interior of \(A\) in \(\tau^*_i(\mathcal{I})\) is denoted by \(\text{Int}^*_i(A)\) and \(\text{Int}^*_i(A^*_i)\) denotes the interior of \(A^*_i\) with respect to topology \(\tau_i\), where \(A^*_i = \{x \in X|U \cap A \notin \mathcal{I}\}\), for every \(U \in \tau_i\). The interior, \(\gamma_i\) of \(A\) is denoted by \(\text{Int}_{\gamma_i}(A)\) and described to be the union of all \(\gamma_i\)-open sets of \(X\) contained in \(A\). The closure, \(\gamma_i\) of \(A\) is denoted by \(\text{Cl}_{\gamma_i}(A)\) and defined to be the intersection of all \(\gamma_i\)-closed sets containing \(A\). Currently, several results and definitions from [2, 3, 7, 13, 17] are recalled to be used in this article.

**Definition 1.** [8] A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) with operation \(\gamma\) on \(\tau_1 \cup \tau_2\) is named:

1. \(\gamma_{ij}\)-semi-open set if \(A \subseteq \text{Cl}_{\gamma_i}(\text{Int}_{\gamma_j}(A))\), where \(i \neq j\) and \(i, j = 1, 2\).
2. \(\gamma_{ij}\)-\(\beta\)-open set if \(A \subseteq \text{Cl}_{\gamma_j}(\text{Int}_{\gamma_i}(\text{Cl}_{\gamma_j}(A)))\), where \(i \neq j\) and \(i, j = 1, 2\).

**Definition 2.** [3] A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) is called

1. \((i, j)\)-semi-\(\mathcal{I}\)-open set if \(A \subseteq \text{Cl}^*_i(\text{Int}_j(A))\), where \(i \neq j\) and \(i, j = 1, 2\).
2. \((i, j)\)-\(\beta\)-\(\mathcal{I}\)-open set if \(A \subseteq \text{Cl}^*_j(\text{Int}_i(\text{Cl}^*_j(A)))\), where \(i \neq j\) and \(i, j = 1, 2\).

**Definition 3.** [16] Let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space with an operation \(\gamma\) on \(\tau_1 \cup \tau_2\). The \(\gamma\)-local function of \(A\) with regard to \(\gamma\) and \(\mathcal{I}\) is described as giving, for \(A \subseteq X\)

\[A^*_\gamma(\gamma, \mathcal{I}) = \{x \in X|U \cap A \notin \mathcal{I}, \text{for every}\ U \in \tau_\gamma(x)\}\]

where \(\tau_\gamma(x) = \{U \in \tau_\gamma|x \in U\}\).

In the case of no ambiguity, we will replace \(A^*_\gamma(\gamma, \mathcal{I})\) by \(A^*_\gamma\).

**Definition 4.** [22] Let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space with an operation \(\gamma\) and \((Y, \sigma_1, \sigma_2)\) be a bitopological space with an operation \(\delta\). Then a function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)\) is called pairwise \((\gamma, \delta)\)-continuous function if \(f^{-1}(V)\) is \(\gamma_i\)-open in \(X\) for all \(\delta_i\)-open set \(V\) in \(Y\), for \(i = 1, 2\).

**Definition 5.** [3] A function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)\) is called to be \((i, j)\)-semi-\(\mathcal{I}\)-continuous function (resp. \((i, j)\)-\(\beta\)-\(\mathcal{I}\)-continuous) if \(f^{-1}(V)\) is \((i, j)\)-semi-\(\mathcal{I}\)-open (resp. \((i, j)\)-\(\beta\)-\(\mathcal{I}\)-open) in \(X\) for all \(\gamma_i\)-open set \(V\) in \(Y\), where \(i \neq j\) and \(i, j = 1, 2\).

Throughout the article, we suppose that \(i \neq j\), and \(i, j = 1, 2\).
3. \( \gamma_{ij} - \beta \mathcal{I} \)-Open Sets

This section deals with the concept of \( \gamma_{ij} - \beta \mathcal{I} \)-open sets and some of their characterizations in an ideal bitopological space.

**Definition 6.** A subset \( A \) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\), with an operation \( \gamma \) on \( \tau_1 \cup \tau_2 \), is said to be \( \gamma_{ij} \)-semi-\( \mathcal{I} \)-open set if \( A \subseteq Cl_{ij}^* (Int_{ij}^*(A)) \).

**Example 1.** Let \( X = \{a, b, c, d\} \) and \((X, \tau_1, \tau_2)\) be a bitopological space with \( \tau_1 = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\} \), \( \tau_2 = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \) and let \( U^\gamma = Cl_j(U) \) for \( U \in \tau_1 \). Then we have, \( \gamma_{12} \)-semi-\( \mathcal{I} \)-open sets are \( \emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\).

**Definition 7.** A subset \( A \) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) is said to be \( \gamma_{ij} - \beta \mathcal{I} \)-open set if \( A \subseteq Cl_{ij}^* (Int_{ij}^*(Cl_{ij}^* (A))) \). The set consisting of all \( \gamma_{ij} - \beta \mathcal{I} \)-open sets in \( X \) will be denoted by \( \gamma_{ij} - \beta \mathcal{I} O(X) \).

**Definition 8.** A subset \( A \) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{I})\) is called \( \gamma_{ij} - \beta \mathcal{I} \)-closed set if the complement \( A^c \) is a \( \gamma_{ij} - \beta \mathcal{I} \)-open set. Equivalently, \( A \) is called \( \gamma_{ij} - \beta \mathcal{I} \)-closed set if \( A \supseteq Int_{ij}^*(Cl_{ij}^* (Int_{ij}^*(A))) \). The set consisting of all \( \gamma_{ij} - \beta \mathcal{I} \)-closed sets in \( X \) will be denoted by \( \gamma_{ij} - \beta \mathcal{I} C(X) \).

**Theorem 1.** Let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space,

(i) Every \( \gamma_{ij} \)-semi-\( \mathcal{I} \)-open set is \( \gamma_{ij} - \beta \mathcal{I} \)-open.

(ii) Every \( \gamma_{ij} - \beta \mathcal{I} \)-open set is \( \gamma_{ij} - \beta \mathcal{I} \)-open.

**Proof.**

(i) Let \( A \) be a subset of \( X \). If \( A \) is \( \gamma_{ij} \)-semi-\( \mathcal{I} \)-open, then

\[
A \subseteq Cl_{ij}^* (Int_{ij}^*(A)) \subseteq Int_{ij}^*(A) \cup (Int_{ij}^*(A))^*_\gamma_j \\
\subseteq (Int_{ij}^*(A)) \cup Cl_{ij}^* (Int_{ij}^*(A)) \subseteq Cl_{ij}^* (Int_{ij}^*(A)) \\
\subseteq Cl_{ij}^* (Int_{ij}^*(A \cup A^*_\gamma_j)) \subseteq Cl_{ij} (Int_{ij}^*(Cl_{ij}^* (A))).
\]

Therefore, \( A \) is a \( \gamma_{ij} - \beta \mathcal{I} \)-open set.

(ii) Let \( A \) be a subset of \( X \). If \( A \) is \( \gamma_{ij} - \beta \mathcal{I} \)-open, then

\[
A \subseteq Cl_{ij} (Int_{ij}^*(Cl_{ij}^* (A))) \subseteq Cl_{ij} (Int_{ij}^*(A^*_\gamma_j \cup A)) \\
\subseteq Cl_{ij} (Int_{ij} (Cl_{ij}^* (A) \cup A)) \subseteq Cl_{ij} (Int_{ij} (Cl_{ij} (A))).
\]

Therefore, \( A \) is a \( \gamma_{ij} - \beta \)-open set.

But generally the convers of this theorem is not true as giving in the next example.
Example 2. From example 1, let \( A = \{c\} \) or \( A = \{b, d\} \). Calculations show that \( A \) is \( \gamma_{12} - \beta I \)-open, however, it is not \( \gamma_{12} - \text{semi-I} \)-open.

Conclusion 1. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. Then every \( \gamma_{ij} - \text{semi-I} \)-open set is \( \gamma_{ij} - \beta I \)-open.

Theorem 2. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space. Then

(i) The union of any \( \gamma_{ij} - \beta I \)-open sets is \( \gamma_{ij} - \beta I \)-open set.

(ii) The intersection of any \( \gamma_{ij} - \beta I \)-closed sets is \( \gamma_{ij} - \beta I \)-closed set.

Proof.

(i) Let \( A_\alpha \in \gamma_{ij} - \beta IO(X) \) for each \( \alpha \in \Lambda \), where \( \Lambda \) is an index set. Then

\[ A_\alpha \subseteq \text{Cl}_{\gamma_j}(\text{Int}_{\gamma_i}(\text{Cl}_{\gamma_j}^* (A_\alpha))). \]

Therefore,

\[
\begin{align*}
\bigcup_{\alpha \in \Lambda} A_\alpha & \subseteq \bigcup_{\alpha \in \Lambda} \left\{ \text{Cl}_{\gamma_j}(\text{Int}_{\gamma_i}(\text{Cl}_{\gamma_j}^* (A_\alpha))) \right\} \\
& \subseteq \left\{ \text{Cl}_{\gamma_j}(\text{Int}_{\gamma_i}(\bigcup_{\alpha \in \Lambda} \text{Cl}_{\gamma_j}^* (A_\alpha))) \right\}
\end{align*}
\]

Then \( \bigcup_{\alpha \in \Lambda} A_\alpha \) is \( \gamma_{ij} - \beta I \)-open.

(ii) The proof follows by using (i) and taking complement.

The intersection of any two \( \gamma_{ij} - \beta I \)-open sets may not be an \( \gamma_{ij} - \beta I \)-open set as shown in the next example.

Example 3. Let \( X = \{a, b, c, d\} \), \( \tau_1 = \{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}\} \), \( \tau_2 = \{\emptyset, X\} \) and \( I = \{\emptyset, \{b\}, \{c\}, \{b, c\}\} \). Let define an operation \( \gamma : \tau_1 \cup \tau_2 \rightarrow P(x) \) such that \( U^\gamma = U \) for all \( U \in \tau_1 \). Then we have \( \{a, c\} \) and \( \{c, d\} \) are \( \gamma_{12} - \beta I \)-open sets but \( \{c\} \) is not \( \gamma_{12} - \beta I \)-open.

Definition 9. Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space with an operation \( \gamma \), \( A \subset X \) and \( x \) be a point of \( X \). Then

(i) \( x \) is called an \( \beta I - \text{interior}_{\gamma_{ij}} \) point of \( A \) if there exists any \( U \in \gamma_{ij} - \beta IO(X) \) such that \( x \in U \subset A \).

(ii) The set of all \( \beta I - \text{interior}_{\gamma_{ij}} \) points of \( A \) is called \( \gamma_{ij} - \beta I \)-interior of \( A \) and is represented by \( \beta I - \text{Int}_{\gamma_{ij}}(A) \).

Theorem 3. Let \( A \) and \( B \) be subsets of \((X, \tau_1, \tau_2, I)\). Then the following properties hold:

1) \( \beta I - \text{Int}_{\gamma_{ij}}(A) = \cup\{ U : U \subset A \text{ and } U \in \gamma_{ij} - \beta IO(X) \} \).
2) $\beta I$-$\text{Int}_{\gamma ij}(A)$ is the largest $\gamma ij$-$\beta I$-open subset of $X$ contained in $A$.

3) $A$ is $\gamma ij$-$\beta I$-open if and only if $A = \beta I$-$\text{Int}_{\gamma ij}(A)$.

The proof will be obtained directly from the definition and thus the proof is omitted.

**Definition 10.** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space with an operation $\gamma$, $A \subset X$ and $x$ be a point of $X$. Then,

1. $x$ is called an $\gamma ij$-$\beta I$-cluster point of $A$ if $U \cap A \neq \emptyset$ for every $U \in \gamma ij$-$\beta I O(X)$ such that $x \in U$.

2. The set of all $\gamma ij$-$\beta I$-cluster points of $A$ is called $\gamma ij$-$\beta I$-cluster of $A$ and is represented by $\beta I$-$\text{Cl}_{\gamma ij}(A)$.

**Theorem 4.** Let $A$ and $B$ be subsets of $(X, \tau_1, \tau_2, I)$. Then the following properties hold:

1) $\beta I$-$\text{Cl}_{\gamma ij}(A) = \cap \{V : A \subset V$ and $V \in \gamma ij$-$\beta I C(A)\}$.

2) $\beta I$-$\text{Cl}_{\gamma ij}(A)$ is the smallest $\gamma ij$-$\beta I$-closed subset of $X$ containing $A$.

3) $A$ is $\gamma ij$-$\beta I$-closed if and only if $A = \beta I$-$\text{Cl}_{\gamma ij}(A)$.

The proof will be obtained directly from the definition and thus the proof is omitted.

**Theorem 5.** Let $(X, \tau_1, \tau_2, I)$ be an ideal bitopological space with an operation $\gamma$ and $A \subset X$. Then,

(i) If $I = \{\emptyset\}$, then $A$ is $\gamma ij$-$\beta I$-open if and only if $A$ is $\gamma ij$-$\beta$-open.

(ii) If $I = P(X)$, then $A$ is $\gamma ij$-$\beta I$-open if and only if $A$ is $\gamma ij$-semi-open.

**Proof.**

(i) We have just to show that if $I = \{\emptyset\}$ and $A$ is $\gamma ij$-$\beta$-open, then $A$ is $\gamma ij$-$\beta I$-open. If $I = \{\emptyset\}$, then $A_{\gamma ij}^* = \text{Cl}_{\gamma ij}(A)$ for all subset $A$ of $X$. Assumed $A$ to be $\gamma ij$-$\beta$-open set, then

$$A \subseteq \text{Cl}_{\gamma ij}(\text{Int}_{\gamma i}(A)) \subseteq \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(A_{\gamma ij}^*))$$

$$\subseteq \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(A_{\gamma ij}^* \cup A)) \subseteq \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(\text{Cl}_{\gamma ij}(A))).$$

Therefore, $A$ is $\gamma ij$-$\beta I$-open.

(ii) Let $I = P(X)$, then $A_{\gamma ij}^* = \{\emptyset\}$ for any subset $A$ of $X$. Let $A$ be $\gamma ij$-semi-open.

Then $A \subseteq \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(A)) = \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(A \cup A_{\gamma ij}^*)) = \text{Cl}_{\gamma i}(\text{Int}_{\gamma \tau i}(\text{Cl}_{\gamma ij}(A))).$ Therefore, $A$ is $\gamma ij$-$\beta I$-open.
Theorem 6. Let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space with an operation \(\gamma\) and \(A \subset X\). Then \(A\) is \(\gamma_{ij}-\beta \mathcal{I}\)-open if and only if \(C\ell_{\gamma_{ij}}(A) = C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}^*(A)))\).

Proof. Let \(A\) be an \(\gamma_{ij}-\beta \mathcal{I}\)-open subset of \(X\). Then \(A \subseteq C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}^*(A)))\). Hence \(C\ell_{\gamma_{ij}}(A) \subseteq C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}^*(A)))\). Since \(A^* \cup A \subseteq C\ell_{\gamma_{ij}}(A)\), then we have,

\[
C\ell_{\gamma_{ij}}(A) \subseteq C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}^*(A))) \subseteq C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}(A))) \subseteq C\ell_{\gamma_{ij}}(A).
\]

Therefore, \(C\ell_{\gamma_{ij}}(A) = C\ell_{\gamma_{ij}}(\text{Int}_{\gamma_{ij}}(C\ell_{\gamma_{ij}}^*(A)))\).

The convers is obvious.

4. \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-Continuous Functions

In this section the concept of \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-continuous function in ideal bitopological spaces are introduced along with some characterizations via related notions.

Throughout this section, let \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space with an operation \(\gamma\), and \((Y, \sigma_1, \sigma_2)\) be a bitopological space with an operation \(\delta\).

Definition 11. A function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)\) is called \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-continuous function (resp. \((\gamma, \delta)_{ij}-\beta \mathcal{I}\) continuous) if \(f^{-1}(V)\) is \(\gamma_{ij}\)-semi-\(\mathcal{I}\)-open (resp. \(\gamma_{ij}\)-\(\beta \mathcal{I}\) open) in \(X\) for all \(\delta\)-open set \(V\) in \(Y\).

Generally every \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-continuous function is \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-continuous, but the convers is not true as giving in next example.

Example 4. Let \(X = \{a, b, c, d\}\) be a set and \((X, \tau_1, \tau_2, \mathcal{I})\) be an ideal bitopological space with

\[
\tau_1 = \{\emptyset, X, \{b\}, \{c, d\}, \{b, c, d\}\},
\]
\[
\tau_2 = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c, d\}\},
\]
\[
\mathcal{I} = \{\emptyset, \{b\}\},
\]

\(U^\gamma = C\ell_{\gamma}^*(U)\) for \(U \in \tau_1\).

Let \(Y = \{p, q, r, s\}\) be a set and \((Y, \sigma_1, \sigma_2)\) be a bitopological space with

\[
\sigma_1 = \{\emptyset, Y, \{q\}, \{r, s\}, \{q, r, s\}\},
\]
\[
\sigma_2 = \{\emptyset, Y, \{p\}, \{p, q\}, \{p, r, s\}\},
\]

\(V^\delta = V\) for \(V \in \sigma_1\).

Let define \(f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)\) such that \(f(a) = p, f(b) = q, f(c) = r\) and \(f(d) = s\). Then \(f\) is \((\gamma, \delta)_{12}-\beta \mathcal{I}\)-continuous but not \((\gamma, \delta)_{12}-\beta \mathcal{I}\)-continuous because \(\{p\}\) is \(\delta\)-open set and \(f^{-1}(\{p\}) = \{a\}\) which is \(\gamma_{12}\)-\(\beta \mathcal{I}\)-open in \(X\) but not \(\gamma_{12}\)-\(\beta \mathcal{I}\)-open in \(X\).

Theorem 7. For any function \(f : (X, \tau_1, \tau_2, \mathcal{I}) \rightarrow (Y, \sigma_1, \sigma_2)\), the next properties are equivalent,

1) \(f\) is \((\gamma, \delta)_{ij}-\beta \mathcal{I}\)-continuous

2) For all \(x \in X\) and every \(\delta\)-open set \(V\) in \(Y\) containing \(f(x)\), there exists a \(\gamma_{ij}-\beta \mathcal{I}\)-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subset V\).
Proof.

(1 \Rightarrow 2) Let V is \(\delta_i\)-open in Y such that \(f(x) \in V\). Since \(f\) is \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous, \(f^{-1}(V)\) is \(\gamma_{ij}\)-\(\beta I\)-open set in X. Let \(U = f^{-1}(V)\). Then \(f(x) \in f(U) \subset V\).

(2 \Rightarrow 1) Let V be \(\delta_i\)-open set in Y and \(x \in f^{-1}(V)\). Then V is \(\delta_i\)-open set in Y and \(f(x) \in V\). From the hypothesis, there exists a \(\gamma_{ij}\)-\(\beta I\)-open set U in X containing x such that \(f(U) \subset V\). Then \(x \in U \in f^{-1}(V)\), i.e. \(f^{-1}(V)\) is \(\gamma_{ij}\)-\(\beta I\)-open set in X. Therefore, \(f\) is \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous.

Theorem 8. Let \(f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)\) be a \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous function. Then the next properties are equivalent:

1) The inverse image of every \(\delta_i\)-closed set in Y is \(\gamma_{ij}\)-\(\beta I\)-closed set in X,

2) \(f(\beta I-Cl_{\gamma_{ij}}(U)) \subset Cl_{\delta_j}(f(U))\), for all subset U of X,

3) \(\beta I-Cl_{\gamma_{ij}}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V))\), for each subset V of Y.

Proof.

(1 \Rightarrow 2) Let \(U \subset X\). Since \(Cl_{\delta_j}(f(U))\) is an \(\delta_i\)-closed set in Y, from the hypothesis, we have \(f^{-1}(Cl_{\delta_j}(f(U)))\) is \(\gamma_{ij}\)-\(\beta I\)-closed set in X. Also \(U \subset f^{-1}(Cl_{\delta_j}(f(U)))\) and \(\beta I-Cl_{\delta_j}(U)\) is the smallest \(\gamma_{ij}\)-\(\beta I\)-closed set containing U. Therefore,

\[
\beta I - Cl_{\delta_i}(U) \subset f^{-1}(Cl_{\delta_j}(f(U))).
\]

This implies that \(f(\beta I-Cl_{\gamma_{ij}}(U)) \subset Cl_{\delta_j}(f(U))\).

(2 \Rightarrow 3) Let \(V \subset Y\). Then \(f^{-1}(V) \subset X\). From the hypothesis,

\[
f(\beta I - Cl_{\gamma_{ij}}(f^{-1}(V))) \subset Cl_{\delta_j}(f(f^{-1}(V))) \subset Cl_{\delta_j}(V).
\]

Hence \(\beta I-Cl_{\gamma_{ij}}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V))\).

(3 \Rightarrow 1) Let \(V\) be a \(\delta_i\)-closed set in Y. From the hypothesis,

\[
\beta I - Cl_{\delta_i}(f^{-1}(V)) \subset f^{-1}(Cl_{\delta_j}(V)) = f^{-1}(V).
\]

Therefore, \(f^{-1}(V) = \beta I-Cl_{\gamma_{ij}}(f^{-1}(V))\) and so \(f^{-1}(V)\) is \(\gamma_{ij}\)-\(\beta I\)-closed set in X.

Theorem 9. The function \(f : (X, \tau_1, \tau_2, I) \to (Y, \sigma_1, \sigma_2)\) is \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous function if and only if

\[
f^{-1}(Int_{\delta_i}(V)) \subset \beta I - Int_{\gamma_{ij}}(f^{-1}(V))
\]

for all \(\delta_i\)-open set of Y.
Proof. Let \( f \) is a \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous function and \( V \) be an \( \delta_i \)-open set in \( Y \). Then \( f^{-1}(\text{Int}_{\delta_i}(V)) \) is a \((\gamma, \delta)_{ij}\)-\(\beta I\)-open set in \( X \). Therefore,

\[
f^{-1}(\text{Int}_{\delta_i}(V)) \subset \beta I - \text{Int}_{\gamma_{ij}} f^{-1}(\text{Int}_{\delta_i}(V)) \subset \beta I - \text{Int}_{\gamma_{ij}} f^{-1}((V)).
\]

If \( f^{-1}(\text{Int}_{\delta_i}(V)) \subset \beta I - \text{Int}_{\gamma_{ij}} f^{-1}((V)) \) and \( V \) be a \( \delta_i \)-open set of \( Y \), then

\[
f^{-1}(V) = f^{-1}(\text{Int}_{\delta_i}(V)) \subset \beta I - \text{Int}_{\gamma_{ij}} f^{-1}((V)).
\]

Therefore, \( f^{-1}(V) \) is \((\gamma, \delta)_{ij}\)-\(\beta I\)-open set in \( X \) and so \( f \) is a \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous function.

Note that, the composition of two \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous functions need not to be \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous, in general.

Example 5. Let \((X, \tau_1, \tau_2, \mathcal{I})\) and \((X, \sigma_1, \sigma_2, \zeta)\) be two ideal bitopological spaces such that \( X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{a, b\}\}, \tau_2 = \{\emptyset, X, \{a, b\}\}, \mathcal{I} = \{\emptyset, \{b\}\} \),

\[
U^\gamma = \begin{cases} \text{Cl}_j(U) & \text{for } U \in \tau_i, \ b \notin U \\ U & \text{for } b \in U \end{cases}
\]

and \( \sigma_1 = \{\emptyset, X, \{b\}, \{b, c\}\}, \sigma_2 = \{\emptyset, X, \{b, c\}\}, \zeta = \{\emptyset, \{a\}\} \), \( V^\delta = V \) for \( V \in \sigma_i \). Let define \( f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2) \) such that \( f(a) = b, f(b) = a \) and \( f(c) = c \) and let \((X, \varsigma_1, \varsigma_2)\) be a bitopological space such that \( \varsigma_1 = \{\emptyset, X, \{a\}, \{c\}\} \), \( \varsigma_2 = \{\emptyset, X, \{b, c\}\} \),

\[
W^\zeta = \begin{cases} \text{Cl}_j(W) & \text{for } W \in \tau_i, \ c \notin W \\ W & \text{for } c \in W \end{cases}
\]

and define \( g : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \varsigma_1, \varsigma_2) \) such that \( g(a) = b, g(b) = c \) and \( g(c) = a \). Then \( f \) is \((\gamma, \delta)_{12}\)-\(\beta I\)-continuous function and \( g \) is \((\delta, \xi)_{12}\)-\(\beta I\)-continuous function but the composition \( g \circ f \) is not \((\gamma, \xi)_{12}\)-\(\beta I\)-continuous function because \( \{a\} \) is \( \xi_1 \)-open set and \( (g \circ f)^{-1}(\{a\}) = \{c\} \notin \gamma_{12}\)-\(\beta IO(X)\).

Definition 12. A function \((X, \tau_1, \tau_2, \mathcal{I}) \to (Y, \sigma_1, \sigma_2)\) is said to be pairwise \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous function if \( f^{-1}(V) \) is \( \gamma_{ij}\)-\(\beta I\)-open in \( X \) for every \( \delta_i \)-open set \( V \) in \( Y \).

Note that the notion of pairwise \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous and \((\gamma, \delta)_{ij}\)-\(\beta I\)-continuous are independent.

Example 6. Let \( X = \{a, b, c\}, \tau_1 = \{\emptyset, X, \{b\}, \{b, c\}\}, \tau_2 = \{\emptyset, X, \{b, c\}\}, \mathcal{I} = \{\emptyset, \{a\}\} \) with operation \( U^\gamma = U \) for \( U \in \tau_i \) and let \((X, \sigma_1, \sigma_2)\) be a bitopological space such that \( \sigma_1 = \{\emptyset, X, \{a\}, \{c\}\}, \sigma_2 = \{\emptyset, X, \{b, c\}\} \) with operation

\[
V^\delta = \begin{cases} \text{Cl}_j(V) & \text{for } V \in \sigma_j, \ c \notin V \\ V & \text{for } c \in V \end{cases}
\]

and define \( f : (X, \tau_1, \tau_2, \mathcal{I}) \to (X, \sigma_1, \sigma_2) \) such that \( f(a) = b, f(b) = c \) and \( f(c) = a \). Then \( f \) is \((\gamma, \delta)_{12}\)-\(\beta I\)-continuous function but it is not \((\gamma, \delta)_{1}\)-\(\beta I\)-continuous function since \( \{a\} \in \delta_1 \)-open and \( f^{-1}(\{a\}) = \{c\} \) which is not \((\gamma, \delta)_{1}\)-\(\beta I\)-open.
Theorem 10. Let $f : (X, \tau_1, \tau_2, I) \rightarrow (Y, \sigma_1, \sigma_2)$ and $g : (X, \tau_1, \tau_2, I) \rightarrow (Z, \ell_1, \ell_2)$. Then $g \circ f$ is $(\gamma, \xi)_{ij}$-$\beta I$-continuous if $f$ is $(\gamma, \delta)_{ij}$-$\beta I$-continuous and $g$ is pairwise $(\delta, \xi)_{ij}$-$\beta I$-continuous.

Proof. Let $W \in \xi_{ij}$-open set in $Z$. Since $g$ is pairwise $(\delta, \xi)_{ij}$-continuous, then $g^{-1}(W) \in \delta_{ij}$-open set in $Y$. On the other hand, since $f$ is $(\gamma, \delta)_{ij}$-$\beta I$-continuous, $f^{-1}(g^{-1}(W)) \in \gamma_{ij}$-$\beta IO(X)$. Therefore, we obtain that $g \circ f$ is $(\gamma, \xi)_{ij}$-$\beta I$-continuous.

5. Conclusion

In this study, we defined the notion of $\gamma_{ij}$-semi-$I$-open sets and $\gamma_{ij}$-$\beta I$-open sets by generalizing $(i, j)$-semi-$I$-open sets and $(ij)$-$\beta I$-open sets in ideal bitopological spaces with an operation $\gamma : \tau \rightarrow P(X)$. We show that every $\gamma_{ij}$-semi-$I$-open set is a $\gamma_{ij}$-$\beta I$-open but the converse is not always true. Then we described the notions $\gamma_{ij}$-$\beta I$-interior and $\gamma_{ij}$-$\beta I$-cluster of a set $A$. Finally we characterized $(\gamma, \delta)_{ij}$-semi-$I$-continuous and $(\gamma, \delta)_{ij}$-$\beta I$-continuous functions and showed that any $(\gamma, \delta)_{ij}$-semi-$I$-continuous function is a $(\gamma, \delta)_{ij}$-$\beta I$-continuous but the converse is not always true. Also it is shown that the composition of two $(\gamma, \delta)_{ij}$-$\beta I$-continuous functions need not to be $(\gamma, \delta)_{ij}$-$\beta I$-continuous. Consequently the following diagrams are true:

$$
\gamma_{ij}$-semi-$I$-open $\longrightarrow \gamma_{ij}$-$\beta I$-open $\longrightarrow \gamma_{ij}$-$\beta$-open

$$
\gamma_{ij}$-$\beta I$-open $\Longleftrightarrow \gamma_{ij}$-$\beta$-open ( $I = \{\emptyset\}$)

$$
\gamma_{ij}$-$\beta I$-open $\Longleftrightarrow \gamma_{ij}$-semi-open ( $I = P(X)$)

$$
(\gamma, \delta)_{ij}$-semi-$I$-continuous $\longrightarrow (\gamma, \delta)_{ij}$-$\beta I$-continuous

These notations, defined in this study, can be extended to other practicable research fields of topology such as fuzzy topology, soft topology, intuitionistic topology and so on. Also generalized separation axioms can be introduced by the concept of generalized $\beta$-open set.

References


REFERENCES


